Mathematical Model of Dynamic Behavior of Poisson Cognitive Systems in the State Space

Alexander Alexandrovich Solodov[1]
A.N. Kosygin Russian State University, 119071, Moscow, Malaya Kaluzhskaya str., 1
aasol@rambler.ru

Keywords: Cognitive system, Random point process, Equation in the state space.

Abstract: The aim of the study is an analytical description of the dynamic behavior of cognitive systems under the influence of events described by random point processes. At the same time, a cognitive system is understood as a technical device that simulates some human reactions, and the behavior of such a system in operation is characterized by a mathematical expectation of the values of the effects processed by the system that accompany the appearance of events at the input of the system. Examples of impacts are given and, for simplicity, an assumption is made about the magnitude of the impact as the amount of information received by the system when an event occurs. Thus, the behavior of the system is identified with the mathematical expectation of the information processed by the current time. To describe the behavior of the system, the theory of Poisson point processes is applied, and the vector of the state of the system is introduced. A new scientific result of the study is a differential equation in the state space of the system for mathematical expectation of the information processed to the current time.

1 INTRODUCTION

Any random phenomenon localized in time can be interpreted as a random point process in which points on the time axis indicate random times of occurrence of events. The most studied class of point processes are Poisson processes and numerous processes generated by Poisson.

Poisson processes describe a wide range of phenomena arising in physics, engineering and in the functioning of cognitive systems. Such processes include, in particular, the transmission of nerve impulses (Lawlor, Perich, Miller, Kording 2018), the formation of droplets in meteorology (Villiers, Schleiss, Veldhuis, Rolf Hut, Giese, 2021), (Villiers, Schleiss, Veldhuis, Rolf Hut, Giese,2019) other phenomena in hydrology and hydroclimatology (Maity, 2018). Poisson processes are used in the development of pixel distribution models in electron microscopy (Meiniel, Olivo-Marin, Angelini 2018), (Zhang, Zhu, Nichols, Wang, Zhang, Smith, Howard, 2018), (Pham,Tran,Phan, Dinh, Ph., Nguyen 2019), imaging of medical tomography (Diwakar, Kumar, 2018), (Zeng, Lv, Huang, 2020). A number of papers are devoted to general methods of image enhancement taking into account the Poisson distribution of signal pixels and noise in images (Bal, Banerjee, Chaki, Sharma, 2020), (Chen, Fu, Wang, Zhai, Wang, Li, 2019), (Kayyar, Jidesh,2018), (Niknejad, Figueiredo, 2018), (Pham, Tran, 2021).

The most complete exposition of the theory of Poisson point processes is contained in (Snyder, Miller, 1991).

The generality of the mathematical description of a wide variety of phenomena allows us to make the assumption that the processing, understanding, and experience of such phenomena can be described in a universal way.

In this paper, the probabilistic behavior of cognitive systems under the influence of Poisson point processes is investigated. For the purposes of this work, cognitive systems will be understood as automatic technical systems that exhibit human-like reactions in some situations.

The following basic assumptions are made in the paper.

At the input of the cognitive system, there is a Poisson point process with a known function of the intensity of the appearance of points.

[1] https://orcid.org/0009-0007-4154-6973

The cognitive properties of the system are taken into account by two circumstances.

Firstly, each point in the cognitive system is assigned a number, which may be random, characterizing the importance of the event for the system. A typical example is the attribution of a certain amount of information to each event, if an information processing system is modeled. Another example is the emotional reaction of a person to the appearance of stress, described in a classic work on psychology (Holmes, Rahe, 1967). In this case, the point is an event that causes stress, and the effects of stress on the system are modeled by the relative magnitude of stress in accordance with the Holmes and Rahe scale (Holmes, Rahe, 1978).

Such Poisson processes are called in (Snyder, Miller, 1991) marked, and the numbers themselves are marks. Everywhere else, for simplicity, it is assumed that the cognitive system processes information, so the mark is identified with the amount of information received by the system.

Secondly, the cognitive system, with its inherent speed, processes, assimilates, adapts to the impact that each event has on it. In this paper, this phenomenon is modeled as the passage of a marked point process through a dynamic system described by stochastic differential equations. Such processes are called filtered point processes.

It is clear that the mathematical expectations of the processes occurring in it are practically a convenient characteristic of the functioning of the system for analysis.

In this regard, the main task to be solved in the work is to obtain differential equations for mathematical expectations by a reworked cognitive information system.

Based on the assumptions made, a stochastic model of the system functioning in time was developed, simple engineering relations and corresponding graphs were obtained.

2 STUDY METHODS

The basis of the research methodology is the assumption that an essential and foreseeable characteristic of the process of functioning of the cognitive system is the mathematical expectation of the value of the processed information system.

In this regard, to obtain the equations for this characteristic, a representation of random processes in the state space of the cognitive system is used, and the equations themselves are derived using the theory of random point processes.

Consider the process of the appearance of points of a random point process developing in continuous time, and assume that the points on the time axis arise in such a way that their position and number are random.

Let the observation begin at time $t_0 = 0$, and after some time t_1 at the current time W_1 a point appeared at the system input, after some other time t_2 at the current time W_2 another point appeared, etc.

Let us introduce into consideration the process N(t) of counting points in which points arose and call it the process of counting points or the counting point process. Thus, the process N(t) is piecewise constant, has unit increments at the moments of the appearance of points W_i and shows how many points appeared in the time interval $[t_0,t)$.

As already mentioned, the times of occurrence of points W_i, and therefore the point intervals t_i and the number of points N(t) are random variables.

Consider an arbitrary time interval [0,t), such that t =T and assume that the number of points that appeared at the points of time t and 0 is equal to N(t) and N(0), respectively. Denote by N(t,0)=N(t)-N(0) the number of points that appeared in this interval, and by P(N(t,s0)=n) the probability that this number of points will be equal to n.

Let us apply the assumption, widely used in various fields of knowledge, that for a small period of time T = Δt, the probability that a point will appear is proportional to some constant with an accuracy infinitely small with respect to Δt and that the probability of two or more points appearing during this time tends to zero:

$$P[N(t,0)=1]= \lambda (t)\Delta t +O(\Delta t), \qquad P[N(t,0]>1= O(\Delta t) \tag{1.1}$$

If we additionally require that the points appear independently of each other, then the distribution of an arbitrary number of points on the interval T is Poisson:

$$P[N(t,0)=n]= = \frac{1}{n!}[\int_0^t \lambda(\tau)d\tau]^n \exp(-\int_0^t \lambda(\tau)d\tau). \tag{1.2}$$

Thus, we will now assume that the point process is a Poisson random point process or simply a Poisson point process in which the times of occurrence of points W 1,W 2, ..., Wi and their number N(t) at time t are random variables.

To carry out calculations related to point processes, the differential of the counting process dN(t) is introduced into consideration, which is determined in an obvious way: dN(t)=1, if a point appeared in a short time interval dt and dN(t)=0 if the point did not appear.

Let us now consider the random point process {N(t), t ≥t 0} of the appearance of points and assign to each i-th point that appears some random variable U i, which we call a mark, and the corresponding process is a Poisson marked process. By definition, these values are external to the point process in the sense that they cannot affect the intensity of the appearance of points. We will assume that each mark of the process U i is equal to the amount of information that the cognitive system received at the corresponding time W i. It is important to emphasize that the value of U i is determined by the cognitive system itself, depending on the purposes for which it was created.

It is natural to assume that the total amount of information received by the cognitive system at time t is equal to

$$U(t)=\sum_{i=1}^{N(t)} U_i \qquad (1.3)$$

Of practical interest will be the mathematical expectation of E[U(t)] of the process U(t). If, for simplicity, we assume that all random variables $U_i$have the same mathematical expectation u, then E[U(t)] is easily calculated using the properties of conditional mathematical expectations and the probability measure (1.1):

$$E[U(t)]= E_N\{E_u[u(t)/N(t)\}= u\int_0^t(\tau)d\tau. \qquad (1.4)$$

Obviously, since the integral is the number of points that appeared on the time interval [0,t), then (1.4) is the sum of the mathematical expectations of the marks that appeared on this interval. From (1.4) follows the differential relation

$$dE[U(t)]= u\lambda(t)dt, \qquad (1.5)$$

which is the methodological basis for further mathematical transformations.

3 RESEARCH RESULTS

Comprehension, processing, mastering of the average amount of information u(t) received and processed by the cognitive system at time t should take some time, and the more time has passed since the information appeared, the less its residual effect on the system. Thus, the problem arises of describing such a property of the cognitive system as inertia. To do this, we will undertake a description of the cognitive system in the form of a linear dynamic system in the state space.

In the most general case, when the appearance of a point is accompanied by attributing to it a random mark from some continuous set $\mathcal{U}$with probability density p(u), the one-dimensional equation of state can be represented as (Snyder, Miller, 1991)

$$Z_R(t)=Z_R(0)+A(t)Z_R(t)dt+$$

$$+\int_{\mathcal{U}} \int_0^t B(t)\, UM(dt, dU). \qquad (2.1)$$

In equation (2.1), Z R(0) is the initial condition, which, based on the essence of the problem, can be assumed to be zero, A(t) and B(t) are known coefficients, $M(dt, dU)$a two-dimensional Poisson process indicating the number of points appearing in the interval dtand having a continuous mark in the intervaldU.

Since we are interested in the mathematical expectations of processes, we will apply the relation for the mathematical expectation of the integral of the process $M(dt, dU)$:

$$E[\int_{\mathcal{U}} \int_0^t M(dt, dU)]=Pr(u\in \mathcal{U}) \int_0^t(\tau)d\tau, \quad (2.2)$$

$Pr(u \in \mathcal{U})$is the probability of the mark hitting the area $\mathcal{U}$.

The integral representation (2.1) implies a stochastic differential equation for the state variable Y(t), in the right part of which there is a two-dimensional Poisson process:

$$dZ_R = A(t)\, Z_R(t)\, dt$$

$$+ \int_{\mathcal{U}} B(t)\, UM(dt, dU),\ Z_R(0) =0. \qquad (2.3)$$

Let's denote the mathematical expectation of the process Z R(t) both by random times of the appearance of points and by random values of marks (assuming, as before, that all marks are discrete $U_i$and their mathematical expectations are the same and equal to u) by Z(t). Then obviously, taking into account the relation (2.2), the differential equation for a non-random process Z(t) will take the form

$$dZ(t) = A(t)Z(t)dt + uB(t)\lambda(t)dt\,, \qquad (2.4)$$

$$Z(0)=0,$$

or taking into account (1.5)

$$dZ(t)=A(t)Z(t)dt+B(t)dE[U(t)], \qquad (2.5)$$

$$Z(0)=0.$$

Thus, in the right part of the differential equation (2.5) there is a differential of the mathematical expectation of the sum of the marks (1.3). The ratio (2.5) has methodological significance and shows how the dynamic characteristics of the cognitive system can be described. Since the exciting dynamic system is the process E[U(t)], the process Z(t) does not indicate the information received into the system that has been processed to the current moment of time, and equations (2.4) and (2.5) determine the behavior of this quantity in time. Note that equation (2.4) is more convenient for theoretical study and calculations.

The differential equation for the information received in the current time in accordance with (1.5) is an obvious modification of equation (2.4) in which A(t) =0 and B(t)=1

$$dY(t) = u\lambda(t)dt\,, \quad z(0)=0, \qquad (2.6)$$

where the notation Y(t) = E[U(t)] is introduced.

Obviously, the amount of information processed by the time t is the difference X(t) of the amount of information received E[U(t)] and the amount of information not processed Z(t)

$$X(t)=Y(t)-Z(t). \qquad (2.7)$$

From the relation (2.7) it follows that

$$dX(t)=dY(t)-dZ(t), \qquad (2.8)$$

Let us introduce into consideration the obvious vector-matrix notation

$$\mathbf{X}(t) = \begin{pmatrix} Z \\ X \end{pmatrix}, \ \mathbf{A}(t) = \begin{pmatrix} A(t) & 0 \\ -A(t) & 0 \end{pmatrix}, \ \mathbf{B}(t) = \begin{pmatrix} B(t)u\lambda(t) \\ 0 \end{pmatrix},$$

with the application of which equations (2.4) and (2.6), taking into account (2.7), take the canonical form:

$$d\mathbf{X}(t) = \mathbf{A}(t)\mathbf{X}(t)dt + \mathbf{B}(t)dt$$
$$\mathbf{X}(0)=0, \qquad (2.9)$$

or in the expanded form

$$d\begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} A(t) & 0 \\ -A(t) & 0 \end{pmatrix}\begin{pmatrix} Z \\ X \end{pmatrix}dt + \begin{pmatrix} B(t)u\lambda(t) \\ 0 \end{pmatrix}dt,$$
$$\begin{pmatrix} Z(0) \\ X(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (2.10)$$

Equations (2.9) are the main scientific result of the work.

4 DISCUSSION OF THE RESULTS OF THE STUDY

To interpret the main results of the study, we apply the general equation (2.10) to the following special case

$$d\begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} 1/\tau & 0 \\ -1/\tau & 0 \end{pmatrix}\begin{pmatrix} Z \\ X \end{pmatrix}dt + \begin{pmatrix} u\lambda \\ 0 \end{pmatrix}dt, \qquad (3.1)$$

It is clear that in equation (3.1) the coefficient A(t)is constant and equal $1/\tau$, and the intensity function of the Poisson process is also constant and equal λ.

Such a choice of parameters allows us to visually describe the reaction of the dynamic system to the input effect in accordance with equation (2.4). Indeed, the process at the output of a dynamic system can be described as

$$Z(t)= u\lambda \int_0^t h(t,s)ds, \qquad (3.2)$$

where the function h(t,s) is called the response function of the system and for the selected coefficients is equal to

$$h(t,s)= \exp(-\frac{t-s}{\tau}), \qquad (3.3)$$

The response function (3.3) is widely used in physics and engineering to describe decreasing processes. In this case, the constant τis a time constant and determines the time scale during which it makes sense to consider the behavior of the response. In more detail, with this choice of coefficients, each point that appears with an average value of information uis processed in accordance with the decreasing exponent (3.3). The typical behavior of the state variable of the dynamical system Z_R is shown in Fig. 1.

In Fig.2. the behavior of the cognitive system for the parameters indicated in the figure is presented,

obtained as a result of calculations in accordance with the ratio (3.1). Calculations are carried out for the normalized time expressed in units of τ, the value of λ is also understood in units of τ. The presented graphs correspond to an intuitive idea of the processing of information (or other impact over time).

The scientific result is a quantitative relationship between the processed information and the input effects on the cognitive system.

5 CONCLUSION

The paper presents the results of applying the theory of random point processes to the study of the dynamic characteristics of cognitive systems described in the state space.

The main result of the work are the equations (2.10), which allow us to calculate the behavior of the cognitive system for any point effects described by a non-stationary function of the intensities of the appearance of points$\lambda(t)$.

The resulting equations are reduced to a canonical form and allow us to apply them to describe hierarchical cognitive structures when the output of one level is the input of another.

The undertaken consideration can be continued in the direction of studying the noise immunity of cognitive systems by introducing appropriate criteria into consideration.

REFERENCES

Bal, A., Banerjee, M., Chaki, R., Sharma, P., 2020. An efficient method for PET image denoising by combining multi-scale transform and non-local means. *Multimedia Tools and Applications*, vol. 79, pp. 29087–29120.

Chen, H., Fu, S., Wang, H., Zhai, L., Wang, F., Li, Y., 2019. Fuzzy singular value shrinkage for Poisson image denoising. *Journal of Modern Optics*, vol. 66, pp. 1569–1578.

de Villiers, D., Schleiss, M., ten Veldhuis, M.-C., Hut, R., van de Giese, N., 2021. Something fishy going on? Evaluating the Poisson hypothesis for rainfall estimation using intervalometers: results from an experiment in Tanzania. *Atmos. Meas. Tech.*, 14, 5607–5623.

Diwakar, M., Kumar, M., 2018. A review on CT image noise and its denoising. *Biomedical Signal Processing and Control*, vol. 42. pages 73–88.

Guyot, A., Pudashine, J., Protat, A., Uijlenhoet, R., Pauwels, V. R. N., Seed, A., Walker, J. P., 2019. Effect ofdisdrometer type on rain drop size distribution characterisation: a new dataset for south-eastern Australia, Hydrol. Earth Syst.Sci., 23, 4737–4761.

Kayyar, S. H., Jidesh, P., 2018. Non-local total variation regularization approach for image restoration under a Poisson degradation. *Journal of Modern Optics*, vol. 65, no. 19, pp. 2231–2242.

Lawlor, P.N., Perich, M.G., Miller, L.E., Kording, K.P., 2018. Linear-nonlinear-time-warppoisson models of neural activity. *J Comput Neurosci*, 45:173–91.

Maity, R., 2018. Statistical Methods in Hydrology and Hydroclimatology, Springer, New York City, New York, USA.

Meiniel, W., Olivo-Marin, J., Angelini, E. D., 2018. Denoising of microscopy images: A review of the state-of-the art, and a new sparsity-based method. IEEE Transactions on Image Processing, vol. 27, no. 8, pp. 3842–3856.

Niknejad, M., Figueiredo, M. A., 2018. T. Poisson image denoising using best linear prediction: a post-processing framework. *European Signal Processing Conference*, pp. 2230–2234.

Pham, C. T., Tran, T. T., 2021. An algorithm for hybrid regularizers based image restoration with Poisson noise. *Kybernetika*, vol. 57, no. 3, pp. 446–447.

Snyder, D.L., Miller, M., 1991. Random Point Processes in Time and Space. 480 p.

Zeng, G. L., Lv, L., Huang, Q., 2020. Poisson-noise weighted filter for time-of-flight positron emission tomography. *Visual Computing for Industry, Biomedicine and Art*, vol. 3, Article number: 10, 4 p.

Zhang, Y., Zhu, Y., Nichols, E., Wang, Q., Zhang, S., Smith, C., Howard, S., 2019. A Poisson — Gaussian denoising dataset with real fluorescence microscopy images. Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (CVPR), pp. 11710–11718.

Zhang, Y., Zhu, Y., Nichols, E., Wang, Q., Zhang, S., Smith, C., Howard, S., 2019. A Poisson — Gaussian denoising dataset with real fluorescence microscopy images. Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (CVPR), pp. 11710–11718.

Holmes, T.H., Rahe, R.H., 1967. The social readjustment rating scale. *Journal of Psychosomatic Research*, Vol. 11, pp. 213-218.

Rahe, R.H., Arthur, R.J., 1978. Life change and illness studies: past history and future directions. *J Human Stress*, 4 (1): 3–15.

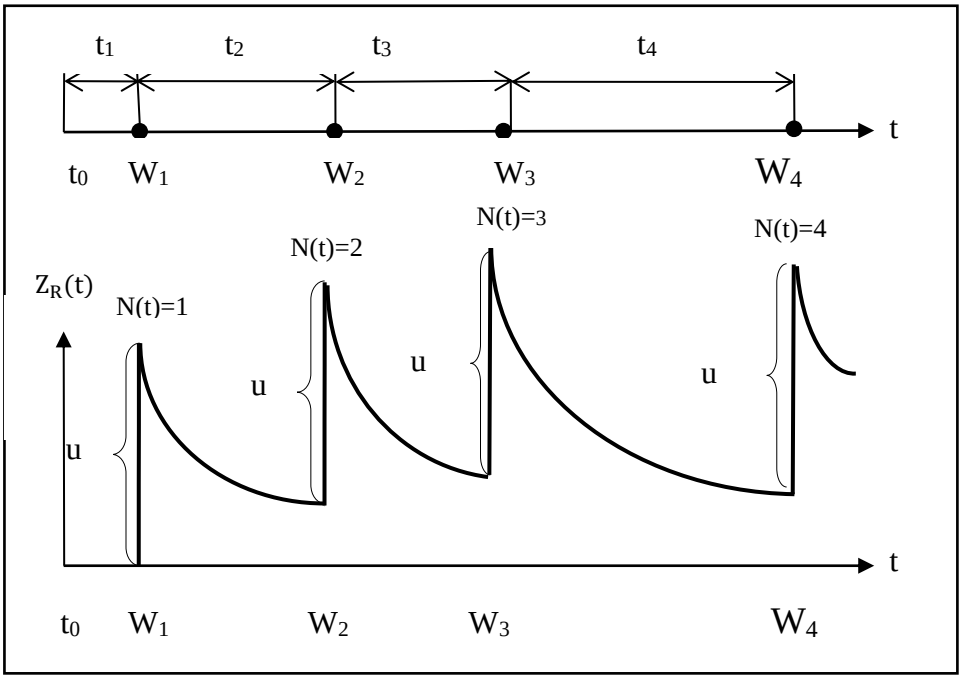
t1
t2
t3
t4
t
t0
W1
W2
W3
W4
ZR(t)
N(t)=1
N(t)=2
N(t)=3
N(t)=4
u
u
u
u
t
t0
W1
W2
W3
W4

Fig. 1

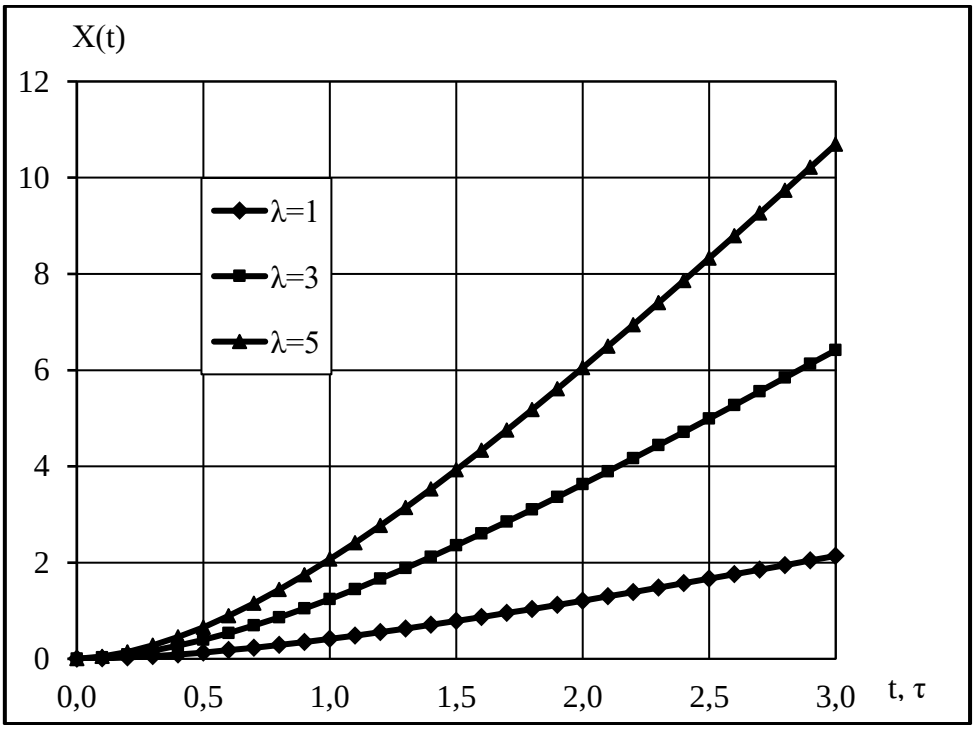
X(t)
12
10
8
6
4
2
0
λ=1
λ=3
λ=5
0,0
0,5
1,0
1,5
2,0
2,5
3,0
t, τ

Fig. 2