On Vibrations of a Moving Elastic Web with Free Fixing Conditions

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Abstract: In this paper, we consider a model initial-boundary value problem of vibrations of a moving elastic web with free fixing conditions. The main task that this article sets for itself is to study and find a solution for the equation of vibrations of an elastic web. For the initial perturbations of a special kind, which are an important part of this problem, exact solutions of this problem are obtained. An algorithm for constructing explicit solutions and a methodology for studying such a class of problems are presented. It is believed that the web moves at a constant speed and it is fixed in a hinged manner. The equation that describes the oscillation process is a fourth-order linear equation with constant coefficients, which contains a term with a mixed derivative. As an illustration, graphs of the web deviation function from the zero position are given. Thus, this article is devoted to the study of vibrations of a moving elastic web with conditions of free fixation.

1 INTRODUCTION

Vibrations of an elastic web are one of the important problems of mathematics and mechanics. For many different branches of science and technology, such as mechanics, aerodynamics, hydrodynamics, electrodynamics and many others, such tasks are of great importance.

Starting to study such a problem as: "Solving a model problem about vibrations of a hinged elastic web", you may encounter a number of interesting questions and tasks. The solution of such a problem allows us to describe the dynamics of deformation and the process of vibrations of an elastic web.

The main difficulty that one has to face when solving this problem is, of course, the complexity of mathematical calculations, which are necessary to determine point solutions.

Another problem may be the fact that it is necessary to take into account many factors that affect the modeling process to one degree or another.

The issues of the study of vibrations, moving materials and structures are relevant and important in modern natural science.

Over the past decades, many studies have been conducted on the subject of vibrations of moving materials.

So it is worth taking into account the relevance of such studies, so the solution of the model problem of vibrations of a hinged elastic web can be applied in practice, and they are widely used in many technical devices.

Equations and some solutions have already been found in the above literature. This paper presents a method for finding new solutions and a universal approach to solving this class of problems.

In [8], important results are presented, for example, such as the question of stability, the values of differential operators are presented, numerical algorithms are constructed, and an analysis of oscillations is carried out.

In [3], the author explores the analysis and control of transverse vibrations of moving strings that move along the axis. The analysis of linear and nonlinear

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string dynamics is carried out, and the influence of string parameters is considered. The paper contains analytical calculations, numerical simulations and experimental results.

Studies of transverse oscillations with a time-dependent velocity of motion are presented in [4], the authors investigate the existence and stability of stationary modes of string oscillations at a time-dependent velocity of motion. The work also includes analysis of transients.

In turn, the work [5] already considers transverse vibrations of the thread at a constant speed. The most important problem considered by the authors is related to the occurrence of resonant effects during transverse vibrations of the thread.

To understand the mathematical methods and calculations that will be made, you can refer to the works [6], [9] and [11], which deals with various sections of mathematics.

In [7], the vibrational behavior of plates consisting of functionally gradient materials is considered under various initial and boundary conditions. Studying the influence of such materials is important.

The considered vibrational behaviors of Timoshenko moving beams in [10] under various boundary conditions help to study the influence of boundary conditions on the vibrational behavior of a moving beam, this is a good help for the development of new effective structures.

The main object of this work is the following equation from [1]:

$$u_{tt} + 2v_0 u_{tx} + (v_0^2 - c^2)u_{xx} + \frac{D}{m}u_{xxxx} = 0. \quad (1)$$

This wave equation gives solutions to transverse vibrations of a moving web. Consider each parameter of this equation: $u = u(x,t)$– this is a function that shows how much the web deviates from the equilibrium point at point x at time t,

v_0 – is the constant velocity at which the wave moves along the x-axis,

c – is the velocity of propagation of vibrations in a stationary web,

D – the parameter that is responsible for bending stiffness,

m – the mass of the web per unit area.

The main problem that needs to be overcome in this equation is that it is not solved by the basic methods for solving differential equations and has very complex mathematical solutions.

If inequality is violated: $v_0 < c$, then this may cause the equation to become unstable. Complex mathematical methods are used to solve such equations.

Below is a new mathematical method for solving this class of equations, with finding several point solutions.

2 STUDY METHODS

Due to the fact that a mixed derivative appears in the equation, it is clearly not possible to apply an ordinary method for solving differential equations, such as the method of separating Fourier variables. Therefore, a new method for solving such equations will be used to find point solutions.

The method is as follows. To begin with, it is necessary to determine in what form a solution will be sought. Next, having defined this function, we substitute it into the original equation and get the form of the equation with the replaced function.

After all sorts of mathematical simplifications, we get an ordinary differential equation. We use the standard method of solving differential equations. We compose a characteristic equation, find the roots of this equation.

We derive solutions for the function we defined earlier and perform a reverse substitution, we get the final form of the solution of the original equation.

After we proceed to solving the problem with initial and boundary conditions, it is necessary to find a basis of exponential functions and perform decomposition on this basis.

From the initial conditions we can determine the coefficients. In this article, based on several examples, you can see the application of the above method.

2.1 Finding a Solution to the Equation

From [1] we have the form of a solution for our equation (1):

$$u(x,t) = \left(C_1 e^{c\lambda\sqrt{1 - \frac{D}{mc^2}\lambda^2}\,t} + C_2 e^{-c\lambda\sqrt{1 - \frac{D}{mc^2}\lambda^2}\,t} \right) e^{\lambda(x - v_0 t)}. \quad (2)$$

Lemma 1. If there is a finite set of values for λ : $\lambda_1, \dots, \lambda_N$, then for each value we have a function that is determined by formula (2) and is the solution of equation (1). And then the linear combination of these solutions will also be a solution and will be determined by the following formula:

$$u(x,t) = \sum_{n=1}^{N} \left(C_{1n} e^{c\lambda_n \sqrt{1-\frac{D}{mc^2}\lambda_n{}^2}\, t} + C_{2n} e^{-c\lambda_n \sqrt{1-\frac{D}{mc^2}\lambda_n{}^2}\, t} \right) e^{\lambda_n (x - v_0 t)}. \quad (3)$$

Next, we will consider the initial boundary value problem for equation (1) with boundary conditions of free fixation:

$$u_{x|_{x=0}} = u_{x|_{x=l}} = u_{xxx|_{x=0}} = u_{xxx|_{x=l}} = 0.$$

Example 1. As a first example, consider the following initial boundary value problem:

$$u_{tt} + 2v_0 u_{tx} + (v_0^2 - c^2) u_{xx} + \frac{D}{m} u_{xxxx} = 0,$$

$$u_{x|_{x=0}} = u_{x|_{x=l}} = u_{xxx|_{x=0}} = u_{xxx|_{x=l}} = 0,$$

$$u|_{t=0} = \cos x, \qquad u_{t|_{t=0}} = 0.$$

In order to represent the existing cosine as a finite set of exponentials, we can write using Euler's formula$\cos x = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$, which allows us to use the ratio (3) with $\lambda_1 = i$, $\lambda_2 = -i$. Next, for a more compact recording, we introduce the parameter $\theta = \sqrt{1 - \frac{D}{mc^2}\lambda_n^2} = \sqrt{1 + \frac{D}{mc^2}}$ для $n = 1, 2.$ Then our desired solution will have the following form:

$$u(x,t) = \left(C_{11} e^{ic\theta t} + C_{21} e^{-ic\theta t}\right) e^{i(x - v_0 t)} + \left(C_{12} e^{-ic\theta t} + C_{22} e^{ic\theta t}\right) e^{-i(x - v_0 t)},$$

In which constants C_{11}, C_{21}, C_{12} и C_{22}can be determined from initial conditions.

So, we have the following relations for the initial conditions of our problem

$$u|_{t=0} = (C_{11} + C_{21}) e^{ix} + (C_{12} + C_{22}) e^{-ix} = \frac{1}{2} e^{ix} + \frac{1}{2} e^{-ix} \qquad (4)$$

and

$$u_t|_{t=0} = \big(C_{11} i (c\theta - v_0) - C_{21} i (c\theta + v_0)\big) \cdot e^{ix} + \big(-C_{12} i (c\theta - v_0) + C_{22} i (c\theta + v_0)\big) \cdot e^{-ix} = 0. \qquad (5)$$

From conditions (4) and (5) we finally obtain two systems for determining the desired constants:

$$\begin{cases} c_{11} + c_{21} = \dfrac{1}{2}, \\ c_{11} i (c\theta - v_0) = c_{21} i (c\theta + v_0), \end{cases}$$

$$\begin{cases} c_{12} + c_{22} = \dfrac{1}{2}, \\ c_{22} i (c\theta + v_0) = c_{12} i (c\theta - v_0). \end{cases}$$

After solving these two systems, we can finally write down the solution for our first example

$$u(x,t) = \frac{1}{2}\left(\left(1 + \frac{v_0}{c\theta}\right) \cos(t(c\theta - v_0) + x) + \left(1 - \frac{v_0}{c\theta}\right) \cos(t(c\theta + v_0) - x)\right).$$

Example 2. Next, as a second example, consider the same initial boundary value problem, but with slightly modified initial conditions:

$$u_{tt} + 2v_0 u_{tx} + (v_0^2 - c^2) u_{xx} + \frac{D}{m} u_{xxxx} = 0,$$

$$u_{x|_{x=0}} = u_{x|_{x=l}} = u_{xxx|_{x=0}} = u_{xxx|_{x=l}} = 0,$$

$$u|_{t=0} = \cos A\, x, \; u_{t|_{t=0}} = \cos B\, x,$$

where $A = \frac{\pi n}{l}$, $n \in \mathbb{N}$and $B = \frac{\pi n'}{l}$, $n' \in \mathbb{N}$.

For the convenience of solving this example, it is necessary to divide the initial problem into two subtasks, and the total solution of this problem will be the sum of the solutions of the two subtasks.

For the first subtask, the initial conditions are set as follows:

$$u|_{t=0} = \cos A\, x, \qquad u_{t|_{t=0}} = 0.$$

Just as in the first example, using Euler's formula we have $\cos \mathrm{A} x = \frac{1}{2}e^{iAx} + \frac{1}{2}e^{-iAx}$, which allows us to use the relation (3) with $\lambda_1 = Ai = \frac{\pi n}{l} i$, $\lambda_2 = -\mathrm{A}i = -\frac{\pi n}{l} i.$ For a more compact recording, we introduce the parameter

$$\theta_A = \sqrt{1 - \frac{D}{mc^2}\lambda_n^2} = \sqrt{1 + \frac{D}{mc^2} A^2} \text{ для } n = 1, 2, \ldots, \infty.$$

Then the desired solution will have the form:

$$u(x,t) = \left(C_{11} e^{Aic\theta_A t} + C_{21} e^{-Aic\theta_A t}\right) e^{Ai(x - v_0 t)} + \left(C_{12} e^{-Aic\theta_A t} + C_{22} e^{Aic\theta_A t}\right) e^{-Ai(x - v_0 t)},$$

Where constants C_{11}, C_{21}, C_{12} и $C_{22}$are defined using the initial conditions for the first subtask.

So, we have the following relations for the initial conditions

$$u|_{t=0} = (C_{11} + C_{21}) e^{Aix} + (C_{12} + C_{22}) e^{-Aix} = \frac{1}{2} e^{Aix} + \frac{1}{2} e^{-Aix} \qquad (6)$$

and

$$u_t|_{t=0} = \big(C_{11}Ai(c\theta_A - v_0) - C_{21}Ai(c\theta_A + v_0)\big)e^{Aix} + \big(-C_{12}Ai(c\theta_A - v_0) + C_{22}Ai(c\theta_A + v_0)\big)e^{-Aix} = 0. \quad (7)$$

From conditions (6) and (7) we can obtain systems for determining constantsC_{11}, C_{21}, C_{12} и C_{22}:

$$\begin{cases} c_{11} + c_{21} = \frac{1}{2}, \\ c_{11}Ai(c\theta_A - v_0) = c_{21}Ai(c\theta_A + v_0), \end{cases}$$

$$\begin{cases} c_{12} + c_{22} = \frac{1}{2}, \\ c_{22}Ai(c\theta_A + v_0) = c_{12}Ai(c\theta_A - v_0). \end{cases}$$

After solving these two systems, we can finally write down the solution for the first subtask

$$u(x,t) = \frac{1}{2}\left(\left(1 + \frac{v_0}{c\theta_A}\right)\cos\big(A(t(c\theta_A - v_0) + x)\big) + \left(1 - \frac{v_0}{c\theta_A}\right)\cos\big(A(t(c\theta_A + v_0) - x)\big)\right).$$

Now, for the second subtask, we will set the initial conditions in a different way, namely this way:

$$u|_{t=0} = 0,\ u_{t|_{t=0}} = \cos B\,x.$$

As in the previous examples, according to the Euler's formula, we have $\cos \mathrm{B}\,x = \frac{1}{2}e^{iBx} + \frac{1}{2}e^{-iBx}$, which allows us to use the ratio (3) with $\lambda_1 = Bi = \frac{\pi n'}{l}i$, $\lambda_2 = -\mathrm{B}i = -\frac{\pi n'}{l}i$.For a more compact notation, we introduce a parameter$\theta_B = \sqrt{1 - \frac{D}{mc^2}\lambda_n^2} = \sqrt{1 + \frac{D}{mc^2}B^2}$ для $n = 1, 2$. And then the desired solution will look like:

$$u(x,t) = \big(C_{11}e^{Bic\theta_B t} + C_{21}e^{-Bic\theta_B t}\big)e^{Bi(x - v_0 t)} + \big(C_{12}e^{-Bic\theta_B t} + C_{22}e^{Bic\theta_B t}\big)e^{-Bi(x - v_0 t)},$$

Where constants C_{11}, C_{21}, C_{12} и C_{22}can also be determined from the initial conditions.

So, we have

$$u|_{t=0} = (C_{11} + C_{21})e^{Bix} + (C_{12} + C_{22})e^{-Bix} = 0 \quad (8)$$

and

$$u_t|_{t=0} = \big(C_{11}Bi(c\theta_B - v0) - C_{21}Bi(c\theta_B + v_0)\big)e^{Bix} + \big(-C_{12}Bi(c\theta_B - v_0) + C_{22}Bi(c\theta_B + v_0)\big)e^{-Bix} = \frac{1}{2}e^{iBx} + \frac{1}{2}e^{-iBx}. \quad (9)$$

From conditions (8) and (9) we obtain systems for determining the desired constants:

$$\begin{cases} c_{11} + c_{21} = 0, \\ c_{11}Bi(c\theta_B - v_0) - c_{21}Bi(c\theta_B + v_0) = \frac{1}{2}, \end{cases}$$

$$\begin{cases} c_{12} + c_{22} = 0, \\ c_{22}Bi(c\theta_B + v_0) - c_{12}Bi(c\theta_B - v_0) = \frac{1}{2} \end{cases}$$

After solving these two systems, we will write down the final solution for the second subtask of the second example we are considering

$$u(x,t) = \frac{1}{2}\left(\frac{1}{Bc\theta_B}\right)\left(\sin\big(B(t(c\theta_B - v_0) + x)\big) + \sin\big(B(t(c\theta_B + v_0) - x)\big)\right).$$

And finally, the general solution for the entire second example will have the following form

$$u(x,t) = \frac{1}{2}\left(\left(1 + \frac{v_0}{c\theta_A}\right)\cos\big(A(t(c\theta_A - v_0) + x)\big) + \left(1 - \frac{v_0}{c\theta_A}\right)\cos\big(A(t(c\theta_A + v_0) - x)\big)\right) + \frac{1}{2}\left(\frac{1}{Bc\theta_B}\right)\left(\sin\big(B(t(c\theta_B - v_0) + x)\big) + \sin\big(B(t(c\theta_B + v_0) - x)\big)\right).$$

Example 3. As a third example, consider this boundary value problem with an initial function $x^4(l - x)^4$.

In order to find a solution to this problem, first it is necessary to decompose this function into a Fourier series by cosines, in order to further represent the cosines obtained during the decomposition as a combination of exponentials, in order to decompose this function into a Fourier series, it is necessary to first familiarize yourself with how to do this, for example, in [2] Pikulin and Pokhazhaev wrote about this in detail, after decomposing the function into a Fourier series, we have the following relation

$$x^4(l - x)^4 = \frac{l^8}{630} + 960\frac{l^8}{\pi^8}\sum_{n=1}^{\infty}\frac{((-1)^n + 1)}{n^8}(\pi^2 n^2 - 42)\cos\frac{\pi n}{l}x.$$

Then we can write down this kind of task

$$u_{tt} + 2v_0 u_{tx} + (v_0^2 - c^2)u_{xx} + \frac{D}{m}u_{xxxx} = 0,$$

$$u_{x|_{x=0}} = u_{x|_{x=l}} = u_{xxx|_{x=0}} = u_{xxx|_{x=l}} = 0,$$

$$u|_{t=0} = x^4(l-x)^4, \qquad u_{t|_{t=0}} = 0.$$

Or, as initial conditions, we write the function in the form of the Fourier series expansion obtained at the previous step

$$u|_{t=0} = \frac{l^8}{630} + 960\frac{l^8}{\pi^8}\sum_{n=1}^{\infty}\frac{((-1)^n+1)}{n^8}(\pi^2 n^2 - 42)\cos\frac{\pi n}{l}x,$$

$$u_{t|_{t=0}} = 0$$

For this solution, we will use the calculations done in Examples 1 and 2. And after identical steps of the solution, we can write down the final form of the solution of this example

$$u(x,t) = \frac{l^8}{630} + 960\frac{l^8}{\pi^8}\sum_{n=1}^{\infty}\frac{((-1)^n+1)}{n^8}(\pi^2 n^2 - 42)\frac{1}{2}\left(\left(1+\frac{v_0}{c\theta_A}\right)\cos\left(\frac{\pi n}{l}(t(c\theta_A - v_0)+x)\right) + \left(1-\frac{v_0}{c\theta_A}\right)\cos\left(\frac{\pi n}{l}(t(c\theta_A + v_0)-x)\right)\right).$$

Example 4. And as the last example, we will consider the previous one, but with different initial conditions

$$u_{tt} + 2v_0 u_{tx} + (v_0^2 - c^2)u_{xx} + \frac{D}{m}u_{xxxx} = 0,$$

$$u_{x|_{x=0}} = u_{x|_{x=l}} = u_{xxx|_{x=0}} = u_{xxx|_{x=l}} = 0,$$

$$u|_{t=0} = 0, \qquad u_{t|_{t=0}} = x^4(l-x)^4.$$

Or

$$u|_{t=0} = 0,\ u_{t|_{t=0}} = \frac{l^8}{630} + 960\frac{l^8}{\pi^8}\sum_{n=1}^{\infty}\frac{((-1)^n+1)}{n^8}(\pi^2 n^2 - 42)\cos\frac{\pi n}{l}x.$$

The final view for this fourth example will look like this

$$u(x,t) = \frac{l^8}{630} + 960\frac{l^8}{\pi^8}\sum_{n=1}^{\infty}\frac{((-1)^n+1)}{n^8}(\pi^2 n^2 - 42)\frac{1}{2}\left(\frac{1}{\frac{\pi n'}{l}c\theta_B}\right)\left(\sin\left(\frac{\pi n'}{l}(t(c\theta_B - v_0)+x)\right) + \sin\left(\frac{\pi n'}{l}(t(c\theta_B + v_0)-x)\right)\right).$$

And finally, we can write down the general solution for example 3 and 4, in which the function acts as the initial function$x^4(l-x)^4$, and it looks like this

$$u(x,t) = \frac{l^8}{315} + 480\frac{l^8}{\pi^8}\sum_{n=1}^{\infty}\frac{((-1)^n+1)}{n^8}(\pi^2 n^2 - 42)\left(\left(\left(1+\frac{v_0}{c\theta_A}\right)\cos\left(\frac{\pi n}{l}(t(c\theta_A - v_0)+x)\right) + \left(1-\frac{v_0}{c\theta_A}\right)\cos\left(\frac{\pi n}{l}(t(c\theta_A + v_0)-x)\right)\right) + \left(\frac{1}{\frac{\pi n'}{l}c\theta_B}\right)\left(\sin\left(\frac{\pi n'}{l}(t(c\theta_B - v_0)+x)\right) + \sin\left(\frac{\pi n'}{l}(t(c\theta_B + v_0)-x)\right)\right)\right).$$

We will also demonstrate this solution on the graph, for the following initial coefficients: $n,\ n' = 300$; $l = 3$; $v_0,\ c = 95$; $m = 10$; $D = 900$. And at times t = 0, 2.9 and 4.4.

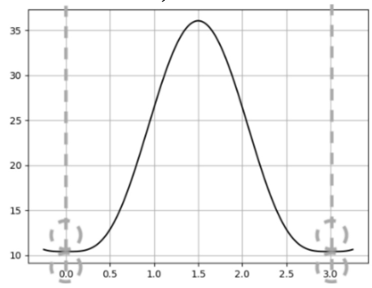

Figure 1 - The graph of the solution at time t = 0

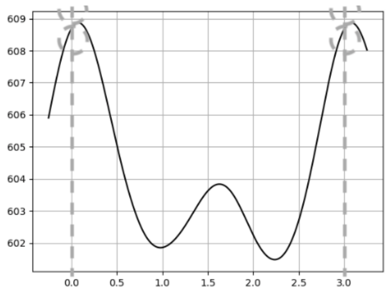

Figure 2 - The graph of the solution at time t = 2.9

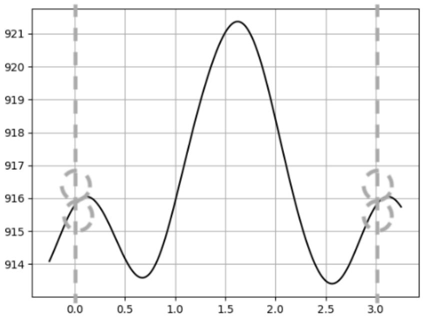

Figure 3 - The graph of the solution at time t = 4.4

3 INVESTIGATION RESULTS

The results of the study show that the vibrations of a moving elastic web with conditions of free fixation manifest themselves in the form of various wave modes. Namely, when the web moves at a constant speed, the formation of waves in front of and behind the web is observed, as well as waves that propagate along the web.

Conducting experiments with changing the speed of movement of the web, it was found that the frequencies of their vibrations also change and the wave itself also changes its shape. In particular, with an increase in speed, a decrease in wavelength is observed, as well as an increase in the frequency of vibrations.

Also changing the stiffness and weight of the web, it was observed that the fluctuations of such a web strongly depend on these parameters. In particular, with an increase in the stiffness of the web or a decrease in its size, an increase in the speed of wave propagation and an increase in the frequency of vibrations is observed.

Modeling of these processes confirms the above judgments. It was found that the simulated solutions of the equation can accurately reproduce the experimental results, which makes it possible to use it for further study of the vibrations of a moving elastic web with free fastening conditions.

4 RESULT DISCUSSION

The results of the study can be said that they attract enormous interest not only from a practical, but also from a theoretical point of view.

On the one hand, they help to better understand the physical processes occurring in a moving elastic web with conditions of free fixation.

On the other hand, they can be useful for the development of new materials and technologies, in which elastic waves play an important role.

One of the most important and basic results of the study is the fact that the oscillatory process of the web depends on its speed, stiffness and size. This dependence makes it possible to determine and refine the parameters necessary for modeling and calculating the behavior of an elastic web in various conditions, for example, when designing and creating structures.

As can be seen from the results of the research, the vibrations of a moving elastic web are a complex phenomenon that depends on many parameters. However, it was possible to describe these fluctuations using a new method.

5 CONCLUSIONS

The paper considers some examples of transverse vibrations of a moving elastic web. As a mathematical model describing this process, an initial boundary value problem for a 4th-order equation with a mixed derivative with constant coefficients is considered. Families of exact solutions are obtained for boundary conditions of free fixation and initial conditions of a special kind. The solution for the initial function, represented as a polynomial of the 8th degree, is given.

Visualization of the solution of the corresponding boundary value problem was also performed at different points in time.

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