Inactive Strategies in the Matrix Game

Assaul V.N.[1][a] and Laschenov V.K.[2][b]
[1]State University of Aerospace Instrumentation; Saint Petersburg, Russia
[2]State University of Sea and River Fleet named after Admiral S.O. Makarov, St. Petersburg, Russia
vicvic21@yandex.ru

Keywords: Matrix game, Lower and upper price of the game, Moves domination, Mixing of strategies, Inactive strategies.

Abstract: Game theory finds many applications in economics, management and other sciences where conflict situations are allowed. The matrix game is a classical linear programming problem, and is well studied. However, it is useful to study the structure and properties of the optimal solution for further use in modified formulations of the problem. The purpose of this article is to study an unconventional method for solving a matrix game that does not rely on the duality theory of linear programming. It is of interest to reduce the dimension of the problem by identifying inactive strategies, that is, moves that are unprofitable for both players.

1 INTRODUCTION

The game-theoretic approach to modeling rivalry situations has proven its effectiveness. Economic and social behavior of subjects (Von Neumann Morgenstern, 1970; Zaharov, 2015; Pecherskij, Belyaeva, 2001; Vasin, Morozov, 2003), military conflicts (Ashkenaz, 1961) are equally well described by the apparatus of game theory. Depending on the nature of the interaction, game models of varying degrees of complexity are used. Such possibilities explain the rapid development of game theory in recent decades.

The zero-sum matrix game is a classic linear programming problem. According to von Neumann's theorem (Von Neumann, Morgenstern, 1970; Owen, 2004), it has a solution in mixed solutions for a payment matrix with non-negative elements. The Danzig algorithm reduces the solution of the matrix game to the consideration of a pair of dual linear programming problems (Owen, 2004; Danzig, 1966).

Previously, if there are negative elements in the payment matrix, a constant is added to all its elements, making the matrix non-negative. This procedure does not change the optimal solutions for both players. Then the reduction of the payment matrix is carried out, crossing out the dominant rows and dominant columns, which reduces the dimension of the problem under consideration (Owen, 2004; Danzig, 1966; Vorob'eva, 1961).

If we expand the concept of dominance to the case of "joint dominance", then we can additionally remove rows and columns that correspond to inactive player strategies. Then they usually solve one of a pair of dual problems by the simplex method, and find the optimal strategies of the players and the price of the game (Zaharov, 2015). At the same time, optimal strategies may also include inactive strategies, that is, rows and columns that, generally speaking, could be crossed out of the payment matrix.

If there are no inactive rows and columns in optimal strategies, then there is no need to apply the simplex method procedure, and the problem is reduced to solving a system of linear equations. Therefore, it is of interest to determine inactive strategies based on the payment matrix. By the way, you can try to solve a system of linear equations, and if it gives positive probabilities for each of the possible moves, then the problem will be solved.

If such a solution is not found, it will indicate the presence of inactive strategies. However, which strategies are inactive will remain unknown.

[a] https://orcid.org/0009-0009-8403-6441
[b] https://orcid.org/0009-0009-3641-3046

2 METHODS AND PRINCIPLES OF RESEARCH

The purpose of this article is to identify inactive strategies without using the simplex method. For this purpose, conditional mathematical expectations of winning players with restrictions in the choice of a move are considered. The properties of the transition graph from one strategy to another are also analyzed.

3 THE MAIN CONTENT OF THE ARTICLE

Let us confine ourselves to the consideration of matrix games with a square matrix, since the optimal solutions correspond to the basic solutions of the linear programming problem (Pecherskij, Belyaeva, 2001), in which the number of non-zero components of the solutions of dual problems coincide.

Let the payment matrix have the form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}. \quad (1)$$

Know,

$\alpha = \max_i a_i = \max_i \min_j a_{ij}, \ i = \overline{1,n}, j = \overline{1,n}$

are called the bottom price of the game, and

$\beta = \min_j \beta_j = \min_j \max_i a_{ij}, \quad i = \overline{1,n}, \ j = \overline{1,n}$

the upper price of the game. The inequality $\alpha \leq \beta$ holds for any payment matrix If $\alpha = \beta$, then the game has a saddle point (Vasin, Morozov, 2003).

Let's generalize the concept of the lower and upper prices of the game to the case when each of the players leaves himself not one, but n-1 move. In other words, we will offer each of the players to give up one of the moves in turn, and we will find the appropriate prices of the game. In fact, these will be conditional mathematical expectations of the price of the game, provided that the players refuse one of the strategies. This makes it possible to build regression curves and analyze them. At the same time, all the lower prices of games will not exceed the upper ones.

It is important to note that the refusal of one of the players from one of their moves allows the second player to take advantage of this. And since crossing out a row entails crossing out a column, and vice versa, the second player maximizes his winnings taking into account the refusal of the first player.

Thus, the refusal of the first player from move i entails the refusal of the second player from move j, and the refusal of the second player from move j entails the refusal of the first player from move k. At the same time, the true price of the game is concluded between the conditional lower and upper prices of the game.

If it turns out that these conditional prices of the game coincided, then it can be argued that both moves that the players refused will be inactive. Indeed, if the first player discards move i, and the second in this case refuses move j and vice versa, then neither of the players can derive any benefit from these moves for themselves, and it makes no sense for both players to use these moves. In this way, inactive moves (strategies) can be detected.

Let's consider an example of a 3x3 matrix game, for which this algorithm is implemented the easiest. Let the payment matrix be given:

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 5 & 3 \\ 2 & 0 & 5 \end{pmatrix}. \quad (2)$$

Let (p 1, p 2, p 3) be the strategy of the first player, and (q 1, q 2, q 3) be the strategy of the second player, with p 1 + p 2 + p 3 = 1, q 1 + q 2 + q 3 = 1. If all the moves of the first player are optimally active, then the system:

$$\begin{cases} 4p_1 + 2p_3 = p_1 + 5p_2 = v \\ 4p_1 + 2p_3 = 2p_1 + 3p_2 + 5p_3 = v, \end{cases}$$

where v is the price of the game, has acceptable solutions.

Indeed, the vector of the optimal strategy of the first player $P^* = (\frac{21}{35}, \frac{13}{35}, \frac{1}{35})$ is the solution of this system of equations, and the vector of the optimal strategy of the second player $Q^* = (\frac{13}{35}, \frac{10}{35}, \frac{22}{35})$ is obtained from the solution of a similar system for the transposed matrix A^T. At the same time, the price of the game is $v = \frac{86}{35}$. As can be seen, in this example there are no inactive rows and columns, and the optimal strategies are found immediately, although the third move is made by the first player much less often than the others, which indicates that the payment matrix (2) is close to the case of inactive strategies.

Let's now consider such a payment matrix:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 5 & 1 \\ 1 & 2 & 2 \end{pmatrix}. \quad (3)$$

Solving a system of equations:

$$\begin{cases} p_1 + 2p_2 + p_3 = 5p_2 + 2p_3 = v \\ p_1 + 2p_2 + p_3 = 3p_1 + p_2 + 2p_3 = v, \end{cases}$$

unlike the previous case, it is not a vector with positive coordinates, therefore, it contains an inactive string. To determine it, we will find the lower and upper prices of the game, provided that the players consistently refuse one of their moves. For the first and second player, the payment matrix changes accordingly as follows:

$$A_1^1 = \begin{pmatrix} 2 & 5 & 1 \\ 1 & 2 & 2 \end{pmatrix}, A_2^1 = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 2 & 2 \end{pmatrix}, A_3^1 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 5 & 1 \end{pmatrix}, A_1^2 = \begin{pmatrix} 0 & 3 \\ 5 & 1 \\ 2 & 2 \end{pmatrix}, A_2^2 = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}, A_3^2 = \begin{pmatrix} 1 & 0 \\ 2 & 5 \\ 1 & 2 \end{pmatrix},$$

here, the upper index indicates the player's number, and the lower number indicates the discarded row or column.

In all cases, it is not difficult to determine the row or column that should be crossed out. In the first matrix, the second column dominates the first and third; in the second, the third over the first and second columns; in the third, sequentially crossing out the columns, you can make sure that deleting the second column gives the minimum price of the game, which means it should be crossed out; in the fourth matrix, the half-sum of the first two rows dominates the third; in the fifth, consecutive crossing out rows gives the maximum price of the game for the third row, and it is crossed out; in the sixth matrix, the second row dominates the first and third. With this in mind, the game prices for the above six matrices will be: α 1=3/2, α 2=1, α 3=5/3, β 1=15/7, β 2=5/3, β 3=2. Figure 1 shows regression curves, i.e. the lower and upper prices of the game depending on the discarded player's move.

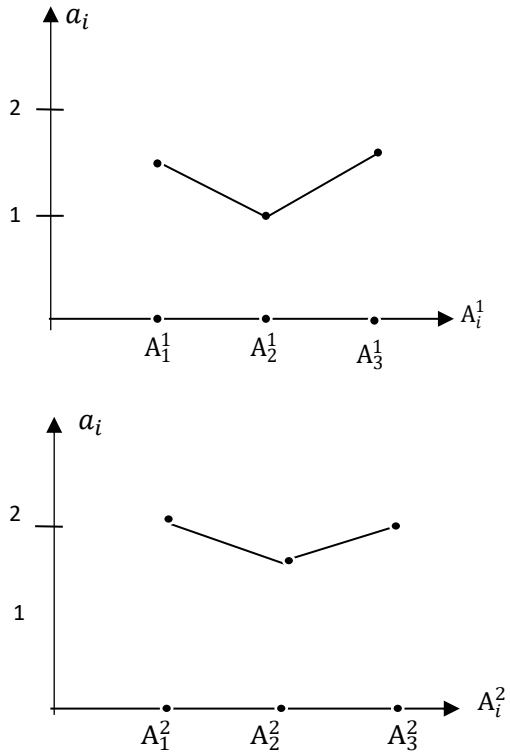

Figure 1: Lower and upper prices for the 3x3 game.

It can be seen that the maximum of the lower and minimum of the upper price of the game coincide. Indeed, max α i = α 3= min β i = β 2 = 5/3, which means that the third row and the second column can be crossed out of the payment matrix (3).

In accordance with what was said above, none of the players can take advantage of the opponent's refusal of these moves. Figure 2 shows a graph showing which column is crossed out due to the deletion of each of the rows, and vice versa, which row is crossed out due to the deletion of columns. The presence of reverse arrows in this column confirms the deletion of the third row and the second column.

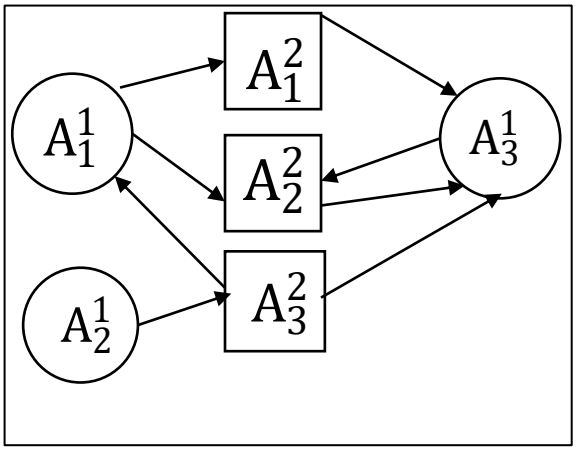

Figure 2: Graph of inactive moves for the 3x3 game.

Let's now consider the fourth-order payment matrix. Obviously, the consideration of smaller submatrices will not be as simple as in the previous case, but the algorithm remains operational.

$$A=\begin{pmatrix} 4\,1\,2\,1 \\ 0\,5\,3\,6 \\ 2\,0\,3\,2 \\ 5\,3\,0\,4 \end{pmatrix}. \qquad (4)$$

It is not difficult to make sure that the system

$$\begin{cases} 4p_1+2p_3+5p_4 = p_1+5p_2+3p_4 = v \\ 4p_1+2p_3+5p_4 = 2p_1+3p_2+3p_3 = v \\ 4p_1+2p_3+5p_4 = p_1+6p_2+2p_3+4p_4 = v \end{cases}$$

does not have strictly positive solutions, which indicates the presence of inactive strategies. Therefore, we apply an algorithm for finding moves that can be excluded from the payment matrix. In this case, for reduced matrices we obtain:

$$A_1^1=\begin{pmatrix} 0\,5\,3\,6 \\ 2\,0\,3\,2 \\ 5\,3\,0\,4 \end{pmatrix}, A_2^1=\begin{pmatrix} 4\,1\,2\,1 \\ 2\,0\,3\,2 \\ 5\,3\,0\,4 \end{pmatrix}, A_3^1=\begin{pmatrix} 4\,1\,2\,1 \\ 0\,5\,3\,6 \\ 5\,3\,0\,4 \end{pmatrix}, A_4^1=\begin{pmatrix} 4\,1\,2\,1 \\ 0\,5\,3\,6 \\ 2\,0\,3\,2 \end{pmatrix},$$

$$A_1^2=\begin{pmatrix} 1\,2\,1 \\ 5\,3\,6 \\ 0\,3\,2 \\ 3\,0\,4 \end{pmatrix}, A_2^2=\begin{pmatrix} 4\,2\,1 \\ 0\,3\,6 \\ 2\,3\,2 \\ 5\,0\,4 \end{pmatrix}, A_3^2=\begin{pmatrix} 4\,1\,1 \\ 0\,5\,6 \\ 2\,0\,2 \\ 5\,3\,4 \end{pmatrix}, A_4^2=\begin{pmatrix} 4\,1\,2 \\ 0\,5\,3 \\ 2\,0\,3 \\ 5\,3\,0 \end{pmatrix}.$$

Consider the first group of matrices $A_1^1 - A_4^1$. In the matrix, A_1^1 the fourth column dominates the second, and therefore is crossed out. In the matrix A_2^1 the first and fourth columns dominate the second, and therefore are crossed out. In matrices $A_3^1 - A_4^1$ the fourth columns dominate over the second, and therefore are crossed out.

Now let's turn to the matrices $A_1^2 - A_4^2$. In the matrix A_1^2 the second row dominates the rest, so rows 1, 3, 4 are crossed out. In the matrix A_3^2 the fourth row dominates the first and third, so the first and third rows are crossed out.

Let 's take a closer look at the matrix A_2^2. There is no obvious dominance here. Let's start crossing out the deadlines in this matrix sequentially. Having crossed out the first line, we get the matrix

$$A_{21}^2=\begin{pmatrix} 0\,3\,6 \\ 2\,3\,2 \\ 5\,0\,4 \end{pmatrix}.$$

Let's try to find optimal strategies, assuming that there is no dominance in the matrix. We obtain a system of equations:

$$\begin{cases} 2p_2+5p_3 = 3p_1+3p_2 = v \\ 2p_2+5p_3 = 6p_1+2p_2+4p_3 = v, \end{cases}$$

which has a solution for the optimal strategy of the first player $P^* = (\frac{21}{35}, \frac{13}{35}, \frac{1}{35})$, while the price of the game v for the matrix A_{21}^2 is 2.47.

Striking out the second row of the matrix A_2^2 results in the matrix

$$A_{22}^2=\begin{pmatrix} 4\,2\,1 \\ 2\,3\,2 \\ 5\,0\,4 \end{pmatrix}.$$

In this matrix, the first column dominates the third, and is crossed out, after which the first row is crossed out. The game price of the remaining second-order matrix is 2.4.

If you cross out the third row in the matrix A_2^2, then in the matrix

$$A_{23}^2 = \begin{pmatrix} 4\ 2\ 1 \\ 0\ 3\ 6 \\ 5\ 0\ 4 \end{pmatrix}$$

by sequentially crossing out rows and columns, you can make sure that crossing out the third row and the third column gives the same price of the game $v = 2.4.$ Finally, crossing out the fourth row of the matrix A_2^2 leads to a matrix

$$A_{24}^2 = \begin{pmatrix} 4\ 2\ 1 \\ 0\ 3\ 6 \\ 2\ 3\ 2 \end{pmatrix},$$

of optimal strategies of the first player for which $P^* = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and the price of the game $v = 2.5$ are found as a solution to a system of equations:

$$\begin{cases} 4p_2 + 2p_3 = 2p_1 + 3p_2 + 3p_3 = v \\ 4p_2 + 2p_3 = p_1 + 6p_2 + 2p_3 = v. \end{cases}$$

Since the highest value of the game price out of all four options is achieved when the last row is crossed out, it should be crossed out in the matrix A_2^2

By performing consecutive strikeouts of rows in the matrix A_4^2, we get that the matrix

$$A_{41}^2 = \begin{pmatrix} 0\ 5\ 3 \\ 2\ 0\ 3 \\ 5\ 3\ 0 \end{pmatrix}$$

has no inactive rows, and the price of its game is 2.214; the matrix

$$A_{42}^2 = \begin{pmatrix} 4\ 1\ 2 \\ 2\ 0\ 3 \\ 5\ 3\ 0 \end{pmatrix}$$

allows the first row and the first column to be crossed out, as can be seen by sequentially crossing out its rows and columns, while the price of the game

$$A_{43}^2 = \begin{pmatrix} 4\ 1\ 2 \\ 0\ 5\ 3 \\ 5\ 3\ 0 \end{pmatrix}$$

is 1.5; the matrix also allows the deletion of the third row and the second column, and the price of its game is 2.4; the matrix

$$A_{44}^2 = \begin{pmatrix} 4\ 1\ 2 \\ 0\ 5\ 3 \\ 2\ 0\ 3 \end{pmatrix}$$

does not contain inactive rows and columns, and its price of the game is 2.421.

The largest of these values $v = 2.421$ determines the row of the matrix A_4^2. to be crossed out. This is the last, fourth row.

Thus, it is possible to create a graph for the payment matrix (4) in the same way as it was done in the previous example. This graph is shown in Fig.3.

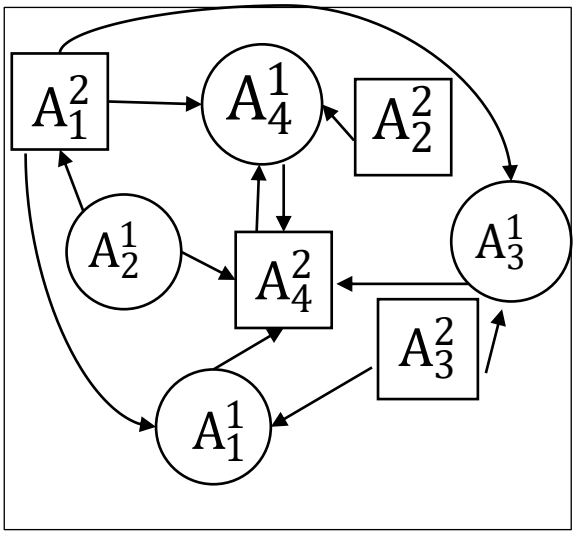

Figure 3: Graph of inactive moves for the 4x4 game.

It follows from it that the matrices A_4^1 and A_4^2 connected by arrows indicate that the fourth row and the fourth column can be crossed out in the original payment matrix. The remaining 3x3 matrix does not contain inactive strategies, and optimal strategies and the price of the game can be easily found by solving the corresponding system of linear equations. Finally we have:

$$P^* = (\tfrac{11}{19}, \tfrac{7}{19}, \tfrac{1}{19}, 0),\ Q^* = (\tfrac{5}{19}, \tfrac{2}{19}, \tfrac{12}{19}, 0),\ v = \tfrac{46}{19}.$$

4 MAIN RESULTS

By expanding the concept of the lower and upper prices of the game, the moves of each of the players are found, the rejection of which makes it possible to find conditional mathematical expectations of winning each of the players, and in case of coincidence of these values to determine the inactive strategies of each of the players. The situation with inactive strategies is associated with the presence of reverse arcs in the graph of inactive moves for two players. The examples given illustrate the proposed algorithm.

5 CONCLUSION

Thus, the construction of a graph of induced strikeout of rows and columns allows you to determine inactive rows and columns in the payment matrix without resorting to the simplex method procedure for a pair of dual linear programming problems. For a problem with a dimension of 4x4 and higher, this may require a larger calculation size, but it allows you to get additional information about the structure of the payment matrix and the properties of subtasks of a smaller dimension for the original problem.

REFERENCES

Von Neumann J., Morgenstern, O., 1970. Teoriya igr i ekonomicheskoe povedenie.

Zaharov, A.V., 2015. Teoriya igr v obshchestvennyh naukah. 304 pages.

Pecherskij, L., Belyaeva, A.A., 2001. Teoriya igr dlya ekonomistov.

Vasin, A.A., Morozov, V.V., 2003. Vvedenie teorii igr s prilozheniyami v ekonomike. 278 pages.

Ashkenaz, V.O., 1961. Primenenie teorii igr v voennom dele.

Owen, G., 2004. Teoriya igr: Per. s angl. Izd. 2-e. 216 pages.

Danzig, Dzh., 1966. Linejnoe programmirovanie, ego primeneniya i obobshcheniya.

Vorob'eva, N.N., 1961. Matrichnye igry.

Assaul, V.N., Lashchenov, V.K., 2020. Mixing Strategies in the Matrix Game. *Journal of Physics: Conference Series (JPCS)*, v 1515.

Kemeni, Dzh., Snell, Dzh., Tompson, Dzh., 1963. Vvedenie v konechnuyu matematiku. 486 pages.