The Influence of Price Constraints on Optimal Resource Allocation

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Keywords: Exchange Economy, Equilibrium, Price and Resource Constraints, Convexity of Preference Relations.

Abstract: This paper examines a model of a closed exchange economy, analyzing various operations undertaken by economic agents. Specifically, it considers scenarios where price constraints are imposed on resource goods, alongside certain constraints on the quantities of exchanged commodities. Such approaches to studying economic processes are motivated by production, economic, and political factors. Typically, and particularly in this work, analysis frameworks in economies of this type enable the identification of a resource subset under natural conditions and ensure that the allocation of goods does not result in an "excessively poor" distribution. In this study, both resource and price constraints are modeled as cones, facilitating their application to economies characterized by growth. The existence of equilibrium, understood in the conventional sense, is proven. Unlike classical cases, this work allows for non-convexities in the preference relations of certain consumer types. It is established that strengthening price and resource constraints enables a greater relaxation of the consumer preference convexity condition.

1 INTRODUCTION

The concept of convexity plays a central role in equilibrium theory (see, for example, (Nikaydo, 1968; Polterovich, 1990; Kantorovich, Katyshev, Kiruta, Polterovich, 1982; Ekeland, 1979)). However, the convexity requirement significantly narrows the scope of research and application of this theory. By attributing convexity to preference relations, we automatically exclude a range of goods from analysis, such as "anti-complementary" goods, indivisible goods, and "mutually exclusive" goods.

Removing the convexity requirement from equilibrium existence proofs presents particular challenges and often necessitates introducing additional constraints into the model. Such conditions, including indivisibility of goods or increasing the number of economic agents, substantially restrict the class of economies under consideration. Studies of this type can be found in the works of McClellan (Mas-Colell, 1977), Yamazaki (Yamazaki, 1978), and Aumann (Aumann, 1966).

This paper explores an alternative approach to relaxing convexity assumptions. In the well-known Arrow-Debreu equilibrium model for an economy with a finite number of goods and consumers (Debreu, 1974.-1975; Debreu, 1972), the convexity condition for preference relations is replaced with a condition of convexity in a cone of directions. Research of this type has been conducted in (Sevodin, Gavrilova, 2011). This paper considers a more generalized case, specifically allowing for resource and price constraints in the model. The existence of equilibrium, understood in the conventional sense, is proven. It is demonstrated that strengthening price and resource constraints enables further relaxation of the consumer preference convexity condition.

2 MODEL DESCRIPTION

Consider an economy in which participants, through exchange, acquire new bundles of goods that best satisfy their needs (see, for example, (Nikaydo, 1968; Ekeland, 1979)). This model does not account for taxes or production.

We assume that consumer behavior is mathematically expressed by the selection of a bundle of goods, represented as a point in the "commodity

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space" $X = (x_1, \dots, x_l) \in Q$, where l - denotes the number of goods available in the market, and the convex cone Q - represents the set of consumer goods. We assume that the cone Q lies within the R_+^l - nonnegative orthant of the l - dimensional real space R^l. Each component of the vector $\mathbf{X}$ – is a nonnegative value $x_k, (k = \overline{1,l})$, indicating the quantity of the respective good.

Let $\mathbf{x}_i \in Q$ denote the portion of resources allocated to the i-th consumer, $i = \overline{1,m}$. Then, an allocation $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_m)$ of resources Ω implies that

$$\Omega = \mathbf{x}_1 + \cdots + \mathbf{x}_m = \sum_{i=1}^{m} \mathbf{x}_i.$$

The problem consists in finding the optimal allocation according to the consumer's preferences. This allocation is characterized as follows: given the resource and price constraints, it will be the best bundle of goods that the consumer can obtain under the specified conditions. The consumer's preference is formalized as a binary relation $\succcurlyeq$ on the set $G \subset Q$, satisfying the properties of reflexivity, transitivity, completeness, and continuity. Additionally, the preference is assumed to be convex and monotonic (see (Nikaydo, 1968; Ekeland, 1979; Sevodin, Gavrilova, 2011)). Under these conditions, system states can be compared, allowing the identification of optimal (equilibrium) states.

However, in certain situations, the convexity requirement is not feasible. Specifically, the convexity requirement implies the convexity of the set G, which is not always the case. We will provide an example illustrating the violation of this property.

Consider the problem of ensuring sufficient cash reserves for a bank's cash desk. The task involves determining the optimal amount of cash across different denominations. The cash reserves should be supplied to the cash desk in volumes that meet the demands of the bank's clients. In general, we have four denominations (100, 500, 1000, and 5000 rubles), which are in demand among the bank's clients.

A distinctive feature of client preferences across different cash denominations is that, in some cases, an increase in the quantity of one denomination leads to a preference against another. This manifests in the following situations. Generally, when withdrawing cash, clients request multiple denominations simultaneously at the cash desk. However, if a client requires only high denominations, they decline smaller denominations, and vice versa. This implies that not all convex combinations of elements in the set G belong to it, meaning G is non-convex.

In this work, we will demonstrate that equilibrium exists even under weaker conditions—monotonicity and convexity in a cone of directions. Let the preference relation possess the following properties (cf. (Sevodin, Gavrilova, 2011)):

α) monotonicity in a non-degenerate cone $K \subset Q: \mathbf{x} + \mathbf{z} \succcurlyeq \mathbf{x}$ for any $\mathbf{x} \in Q$, and for any $\mathbf{z} \in K$, and moreover, there exists a number $N > 0$, such that the relation $\mathbf{x} + \mathbf{z} \succcurlyeq \mathbf{x}$ holds for any $\mathbf{x} \in Q$, $\mathbf{z} \in K$, $\|\mathbf{z}\| \geq N$;

β) Strict convexity in a cone of directions T: for all $\lambda \in [0,1]$ and for all $\mathbf{x}, \mathbf{y} \in Q$ the following property holds:

$$\mathbf{y} \succcurlyeq \mathbf{x}, \mathbf{y} \neq \mathbf{x}, \mathbf{y} - \mathbf{x} \in T, \text{hence } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \succ \mathbf{x}.$$

In these definitions, $K \subset R_+^l$ – represents any non-degenerate cone that includes as an interior point the point with unit coordinates, and the cone T is the closure of the set $R^l \setminus (\pm V)$, V – some cone, $V \subset R_+^l$.

The notation $\mathbf{x} \succ \mathbf{y}$, as usual, denotes strict preference, meaning a situation where $\mathbf{x} \succcurlyeq \mathbf{y}$, but $\mathbf{y} \succcurlyeq \mathbf{x}$ is not applicable ($\mathbf{x}, \mathbf{y} \in Q$).

3 EXISTENCE OF EQUILIBRIUM

The preference relation can be described using a utility function $u(\mathbf{x})$. A real-valued function $u(\mathbf{x})$, defined on G, is called a utility function representing the preference relation $\succcurlyeq$, if, for any $\mathbf{x}, \mathbf{y} \in G$ the condition $u(\mathbf{x}) \geq u(\mathbf{y})$ holds if and only if $\mathbf{x} \succcurlyeq \mathbf{y}$.

Let us identify a class of functions that would define preference relations that are convex in the cone of directions T.

The function $u: G \to R$ will be called quasi-concave in the cone of directions T, if, for any $a \in R$ the set

$$S_a = \{\mathbf{z} \in G \mid u(\mathbf{z}) \geq a\}$$

is convex in the cone of directions T.

The following holds:

Theorem 1 (Sevodin, Gavrilova, 2011**).** A preference relation is convex in the cone of directions T if and only if all of its representing utility functions are quasi-concave in the cone of directions T.

For the case of convex preferences, this theorem was proven in (Ekeland, 1979; Sevodin, Gavrilova, 2011). In our case, the proof is analogous.

Let participant i be characterized by their preference relation $\succcurlyeq_i$ and initial resources $\boldsymbol{\omega}_i \in R_+^l$. Their utility function u_i depends solely on the bundle of goods allocated to participant i. We also assume that the preference relations $\succcurlyeq_i$, $i = \overline{1,m}$, satisfy properties $\alpha), \beta)$.

We introduce a system of prices for goods $\mathbf{p} = (p_1, \dots, p_l) \in P(\subset \operatorname{int} R_+^l)$, where P is the convex, conical hull of certain linearly independent vectors $\mathbf{p}_1, \dots, \mathbf{p}_l \in R_+^l$.

It is worth noting that cone-type constraints applied to both resource and price vectors in this work naturally arise in many economic problems. Examples include situations where proportional increases in resources for technological processes result in a narrowing of the production set, motivated by production, economic, or political factors. In such cases, the resource vector belongs to a convex, closed cone within the nonnegative orthant of the space, thereby introducing cone-type constraints. Furthermore, classical equilibrium models assume the flexibility to choose any nonnegative prices for production resources. However, this assumption does not hold in practice, necessitating constraints on the goods produced and consumed, as well as on their corresponding prices, to balance supply and demand. Studies, such as those on expanding economies of the Neumann type (Nikaydo, 1968), where price changes form a trajectory, lead to the analysis of economies with constraints in the form of certain cones.

Since prices can be normalized (Nikaydo, 1968), we will consider prices on the simplex

$$\overline{\Pi} = \left\{ \mathbf{p} \in P \middle| \ \mathbf{p} = \sum_{k=1}^{l} \alpha_k \mathbf{p}_k, \sum_{k=1}^{l} \alpha_k = 1, \alpha_k \geq 0 \right\}.$$

Furthermore, we assume that the coordinates of the vector $\mathbf{p}$ are positive and sum to one.

Let a participant i be able to choose any bundle $\mathbf{y} \in Q$ from the budget set

$$I(r_i) = \left\{ \mathbf{y} \in Q \middle| \ \langle \mathbf{p}, \mathbf{y} \rangle \leq r_i = \sum_{k=1}^{l} p_k \omega_k^i, \mathbf{p} \in \Pi \right\}.$$

where $\boldsymbol{\omega}_i = \left(\omega_1^i, \dots, \omega_l^i\right) \in Q$.

Participant i acts according to their preference relation $\succcurlyeq_i$, hence choosing a bundle of goods $\mathbf{x}_i \in I(r_i)$ such that $\mathbf{x}_i \succcurlyeq_i \mathbf{y}$ for any $\mathbf{y} \in I(r_i)$. Therefore, further analysis of the model involves finding a price system $\mathbf{p}$, under which an admissible allocation $X = (\mathbf{x}_1, \dots, \mathbf{x}_m)$ consists of vectors $\mathbf{x}_i$, that solve the maximization problem u_i subject to the corresponding budget constraint. We present a statement generalizing the well-known theorem (Ekeland, 1979) on the existence of such an allocation.

Let $\Pi^{\perp}$ denote the cone composed of all vectors $\mathbf{x} \in R^l$, for which there exists a vector $\mathbf{p} \in \Pi$, that $\langle \mathbf{x}, \mathbf{p} \rangle = 0$.

Theorem 2. Let the exchange economy satisfy the above conditions and let the equality $T = \Pi^{\perp}$ hold. Then, for each i and each $r_i \geq 0$ there exists a unique vector $\mathbf{x}_i \in I(r_i)$ such that $\mathbf{x}_i \succcurlyeq_i \mathbf{y}$ for any $\mathbf{y} \in I(r_i)$.

Proof. Given the requirements imposed on the preference relation, the utility function u_i attains its maximum value on the compact set $I(r_i)$ at some point $\mathbf{x}_i$ (Ekeland, 1979). This point belongs to the set $I(r_i)$ and, furthermore, satisfies the equality $\langle \mathbf{p}, \mathbf{x}_i \rangle = r_i$. It remains to show that this point is unique.

Suppose there exist two points $\mathbf{x}_i \in I(r_i)$ and $\mathbf{z}_i \in I(r_i)$, $\langle \mathbf{p}, \mathbf{x}_i \rangle = \langle \mathbf{p}, \mathbf{z}_i \rangle = r_i$, that satisfy the condition: $\mathbf{x}_i \succcurlyeq_i \mathbf{y}$ and $\mathbf{z}_i \succcurlyeq_i \mathbf{y}$ for any $\mathbf{y} \in I(r_i)$.

By sequentially setting $\mathbf{y} = \mathbf{x}_i$ and $\mathbf{y} = \mathbf{z}_i$ in these relations, we obtain that $\mathbf{x}_i \sim \mathbf{z}_i$.

Now, let's confirm that $\mathbf{x}_i - \mathbf{z}_i \in T$. But $\langle \mathbf{p}, \mathbf{x}_i \rangle = \langle \mathbf{p}, \mathbf{z}_i \rangle = r_i$, and thus $\langle \mathbf{p}, \mathbf{x}_i - \mathbf{z}_i \rangle = 0$. Since $\mathbf{p} \in \Pi$, it follows from the last equality that $\mathbf{x}_i - \mathbf{z}_i \in T$.

Due to the strict convexity of the preference relation $\succ_i$ in the cone of directions T, it follows that

$$\left(\frac{1}{2}\mathbf{x}_i + \frac{1}{2}\mathbf{z}_i\right) \succ_i \mathbf{x}_i. \qquad (1)$$

The point $\left(\frac{1}{2}\mathbf{x}_i + \frac{1}{2}\mathbf{z}_i\right)$ belongs to $I(r_i)$, since this set is convex; therefore, equation (1) contradicts the maximality of $\mathbf{x}_i$. This means that the assumption j of the existence of points $\mathbf{x}_i$ and $\mathbf{z}_i$ is incorrect. Theorem 2 is proven.

Let's introduce the individual demand function for participant i. By individual demand, we mean the mapping $\mathbf{d}_i$, that associates any price system $\mathbf{p} \in \Pi$ with the vector $\mathbf{x}_i$ from the participant's budget set, which is preferred. In (Sevodin, Gavrilova, 2011), it is proven that for any i the mapping $\mathbf{d}_i: \Pi \to R_+^l$ is continuous and satisfies the equality

$$\langle \mathbf{p}, \mathbf{d}_i(\mathbf{p}) \rangle = \langle \mathbf{p}, \boldsymbol{\omega}_i \rangle.$$

Examining the behavior of the mapping $\mathbf{d}_i: \Pi \to R_+^l$ near the boundary points Π, we conclude that the

scenario established in (Sevodin, Gavrilova, 2011) remains unchanged, despite the loss of convexity and the presence of price constraints. For example, if participant i is not satiated with good k, their consumption of good k_0 will grow without bound as the price

$$\mathbf{p} = \sum_{k=1}^{l} \alpha_k \mathbf{p}_k$$

approaches the price represented by $\alpha_{k_0} = 0$. The reason is that in this case, the scalar representation $\langle \mathbf{p}, \mathbf{x} \rangle$ lacks the term with α_{k_0}, allowing the vector $\mathbf{p}$ to vary freely in the direction of vector $\mathbf{p}_{k_0}$.

Let's introduce the excess demand function $\mathbf{z}: \Pi \to R_+^l$ (Sevodin, Gavrilova, 2011);

$$\mathbf{z}(\mathbf{p}) = \sum_{i=1}^{m} \mathbf{d}_i(\mathbf{p}) - \mathbf{\Omega}.$$

Let's express the equality of supply and demand in the market using the equation R^l:

$$\mathbf{z}(\mathbf{p}) = 0. \qquad (2)$$

If the price vector $\mathbf{p} \in \Pi$ satisfies condition (2), it is called an equilibrium price system, and the vector $(\mathbf{d}_1(\mathbf{p}), \dots, \mathbf{d}_m(\mathbf{p}))$ is called an equilibrium allocation.

We state the theorem proving the existence of equilibrium prices for the economic model under consideration. Given the above price and resource constraints, we have:

Theorem 3. Let the private ownership economy be such that all m participants have preference relations with properties $\alpha), \beta)$. If initially each participant possesses all goods, and $\boldsymbol{\omega}_i = (\omega_1^i, \dots, \omega_l^i) \in Q, \quad i = \overline{1,l}$, then there exists at least one equilibrium price system or there exists a vector $\mathbf{q} \in \Pi$ such that $\mathbf{z}(\mathbf{q}) = 0$.

Proof. We will prove the theorem by modifying the proof of a similar theorem from (Ekeland, 1979). Let us denote the set of possible coefficients in the representation of Π through $\overline{S}$, i.e.

$$\overline{S} = \{\boldsymbol{\alpha} = \boldsymbol{\alpha}(\mathbf{p}) = (\alpha_1, \dots, \alpha_l) | \sum_{k=1}^{l} \alpha_k = 1, \alpha_k > 0\}$$

Let $S = \operatorname{int} \overline{S}, S_k = \{\boldsymbol{\alpha} \in \overline{S} \mid \alpha_k = 0\}$, as well.

The proof of the theorem is based on the existence of a fixed point of the multivalued mapping F of the simplex $\overline{\Pi}$ onto itself, which is defined as follows (Ekeland, 1979):

$$F(\mathbf{p}) = \{\overline{\Pi}_{K(\mathbf{p})}, \boldsymbol{\alpha}(\mathbf{p}) \in S;\ \mathbf{q} \in \overline{\Pi} \mid \langle \boldsymbol{\alpha}(\mathbf{p}), \boldsymbol{\alpha}(\mathbf{q}) \rangle = 0, \mathbf{q} \notin \overline{\Pi}\}.$$

Here,

$$\overline{\Pi}_{K(\mathbf{p})} = \{q \in \Pi |\ \forall k \notin I(\mathbf{p}):\ q_k = 0\},$$

$$I(\mathbf{p}) = \left\{k \mid z_k(\mathbf{p}) = \max_j z_j(\mathbf{p})\right\}.$$

Due to the imposed conditions, the mapping F has values that are convex, closed, and non-empty sets. This is sufficient to conduct the proof of the theorem similarly to (Sevodin, Gavrilova, 2011), using.

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