10.63550/ICEIP.2026.74.22.061

An Algorithm for Inverting Characteristic Functions in Risk Insurance Problems Using Distribution Moments

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Keywords: risk insurance, insured damage, distribution function, discount factor, numerical methods

Abstract: The paper proposes an algorithm for inverting the characteristic functions obtained when solving actuarial problems of risk insurance. The algorithm is based on calculating the sequence of moments of the desired distribution, which coincides with the desired distribution in the sense of a set of moments. From a technical point of view, the algorithm is implemented by performing a number of linear algebra procedures.

1 INTRODUCTION

In a significant number of risk insurance tasks, it is advisable to calculate the total insured damage in terms of characteristic functions. The preference for using the apparatus of characteristic functions is due, firstly, to the simplicity of implementing the operation of summing independent random variables, which is associated with a rather cumbersome operation of collapsing the distributions of terms (in terms of characteristic functions, this operation is reduced to multiplying the characteristic functions of terms), and secondly, to the need to take into account the discount factor in a number of tasks of risk insurance, ensuring bringing lump sum payments (receipts) to a single - Taking into account the influence of the discount factor in terms of characteristic functions turns out to be significantly simpler than in terms of distribution functions.

In a number of tasks solved using the accumulation model to describe the total insured damage caused to objects (persons) insured in a given company, the use of characteristic functions allows us to obtain a solution in a closed form. It can be expected that in actuarial tasks of long-term life insurance, an integral feature of which is the need to take into account the discount factor, the use of characteristic functions will be very effective. Unfortunately, on the way to practical solution of actuarial problems of risk insurance in terms of characteristic functions, there is a difficulty associated with the conversion of the obtained characteristic function, that is, finding a distribution function that has as a characteristic function the characteristic function obtained in the process of solving the problem.

Since the expression of the characteristic function obtained by solving the problem itself often turns out to be set algorithmically, that is, in the form of a computational algorithm implemented by computer means, and, consequently, the conversion procedure cannot be performed analytically, we can only talk about numerical methods of inverting characteristic functions. Here it is necessary to make a reservation regarding the formula of reversal of characteristic functions known in probability theory. Using the inversion formula to determine the distribution function corresponding to a given characteristic function is very, very difficult. The use of well-known quadrature formulas of the Stieltjes integral, or algorithms of the Riemann integral built into the software package when implementing the conversion formula, leads to the appearance of high-frequency components as a result of calculations that cannot be physically interpreted and do not allow practical use of the results of the conversion procedure. The situation would be hopeless and the application of the apparatus of characteristic functions would remain nothing more than a purely theoretical scheme for

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solving problems if the characteristic function found in a procedural and algorithmic form did not make it relatively easy to calculate moments of any order of the desired distribution function. Here the following fundamental possibility arises for calculating the distribution function corresponding to a given characteristic function: on the one hand, the characteristic function uniquely determines the values of moments of any order.

$$\mu_r = \frac{1}{i^r} \lim_{u \to 0} \varphi^{(r)}(u) \ (r = 0, 1, 2 \dots) \qquad (1)$$

On the other hand, the sequence of moments $\mu_r \ (r = 0, 1, 2 \dots)$, as proved in the theory of mathematical statistics, determines the distribution function in the only way F(x). The corresponding theorem states that if for some $c > 0$ series $\sum_{r=0}^{\infty} \frac{\mu_r}{r!} \cdot c^r$ converges absolutely, then the set of moments $\mu_r \ (r = 0, 1, 2 \dots)$, corresponds to a single distribution function F(x). Unfortunately, the theory of this issue only asserts the existence of a single distribution function corresponding to a given sequence of moments, but does not provide recommendations on how to find this distribution function. Nevertheless, it follows from the theorem that if, using one procedure or another, it is possible to construct a certain distribution function F(x) using a set of moments $\mu_r \ (r = 0, 1, 2 \dots)$, then this distribution function is, firstly, the only one, and secondly, it is a reversal of the characteristic functions $\varphi(u)$.

Thus, the task of inverting the characteristic function $\varphi(u)$ was reduced to constructing a distribution function F(x), the moments of which are equal to $\mu_r \ (r = 0, 1, 2 \dots)$. This paper describes the procedure for constructing a distribution function based on the totality of distribution moments, assuming that this set of moments is determined using the characteristic function, and thus, is devoted to the practical solution of the problem of addressing a certain class of characteristic functions that arise in the tasks of risk insurance.

2 AN ALGORITHM FOR CONSTRUCTING A DISTRIBUTION FUNCTION BASED ON A SET OF MOMENTS

To start calculating the distribution function over a set of moments $\mu_r \ (r = 0, 1, 2 \dots)$, we assume that it is possible to represent the desired distribution function as a series:

$$F(x) = p_0 h(x) + \sum_{k=1}^{\infty} p_k F_0^{(k)}(x), \qquad (2)$$

where $F_0^{(k)}(x) = k -$ multiple convolution of some distribution function $F_0(x)$;

$p_k (k = 0, 1, 2 \dots) -$ weighting factors to be determined;

$h(x)$ – a single jump function (Heaviside function) of the form

$$h(x) = \begin{Bmatrix} 0 \ if \ x \le 0; \\ 1 \ if \ x > 0 \end{Bmatrix}. \qquad (3)$$

The choice of the distribution function $F_0(x)$ and the coefficients $p_k (k = 0, 1, 2 \dots)$ is based on the condition that the moments of the function (2) coincide with the moments (1) of the invertible characteristic function $\varphi(u)$.

Note here that the structure of the approximating function (2) is quite similar to the well-known accumulation model, however, unlike the accumulation model, the distribution function $F_0(x)$ will not be interpreted as a function of the distribution of insured damage in a single insured event, respectively, the coefficients $p_k (k = 0, 1, 2 \dots)$ will not be interpreted as the probability of occurrence of k insurance events.

Thus, expression (2) has only a formal resemblance to the accumulation model and was adopted for reasons of sufficiency of "degrees of freedom" to ensure the identity of the moments of the initial characteristic function and the approximating distribution.

Since the choice of the distribution function $F_0(x)$ is not constrained by any restrictions, except for the natural distribution $F_0^{(k)}(x)$ $p_k (k = 0, 1, 2 \dots)$, and the corresponding moments of these convolutions. These considerations naturally lead to the choice of the following as the $F_0(x)$ gamma distribution

$$F_0(x) = F_\gamma(x; \alpha, \beta), \qquad (4)$$

where $F_\gamma(x; \alpha, \beta)$ – the gamma distribution function defined by the expressions

$$F_\gamma(x; \alpha, \beta) = \int_0^x p_\gamma(y; \alpha, \beta) dy;$$

$$p_\gamma(y; \alpha, \beta) = \begin{cases} 0 \ if \ y \le 0; \\ \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha - 1} e^{-\beta y} \ if \ y \le 0;' \end{cases} \qquad (5)$$

where $p_\gamma(y; \alpha, \beta)$ – the density of the gamma distribution ($\alpha > 0$; $\beta > 0$).

The choice of a distribution function of the form (4), firstly, due to the well-known properties of gamma distributions, makes it easy to find convolutions of any multiplicity, implemented using standard (built-in) functions of well-known software and mathematical packages

$$F_0^{(k)} = F_\gamma(x; k\alpha, \beta), \quad (6)$$

Secondly, it makes it easy to calculate moments of any order

$$\mu'_{rk} = \int_0^\infty x^r p_\gamma(x; k\alpha, \beta)dx = \\ = \int_0^\infty x^r \frac{\beta^{k\alpha}}{\Gamma(k\alpha)} x^{k\alpha-1}dx = \\ = \frac{\beta^{-\gamma}\Gamma(k\alpha + r)}{\Gamma(k\alpha)} \ (r = 0, 1, 2 \ldots) \quad (7)$$

where $\Gamma(\alpha)$ is a gamma function defined by an Euler integral of the form $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz$.

Using the introduced notation, we reformulate the task set above. To this end, we will assign a vector of its moments to each distribution function $F_y(x; k\alpha, \beta)$, so that the r-th coordinate of this vector is expressed by formula (7).

Further, by virtue of formula (2), the r-th moment of the approximating function is determined by the expression

$$\mu_r^{(\alpha)} = \int_0^\infty x^r d\left[p_0 h(x) + \sum_{k=1}^\infty p_k F_0^{(k)}(x)\right] \quad (8)$$

$$= \sum_{k=1}^\infty p_k v_{\gamma rk}$$

where $v_{\gamma rk} = \mu'_{rk}$.

Now let's make up a system of equations defining unknown parameters p_k $(k = 1, 2 \ldots)$. Note that the value of the weighting coefficient for the Heaviside function p_0 is equal to a constant in the expression of the characteristic function (in particular, if there is no such constant, then $p_0 = 0$.

Equating the expressions of the moments of the approximating distribution (8) to the values of the moments of the initial characteristic function (1), we obtain a system of linear equations with respect to the parameters p_k $(k = 1, 2 \ldots)$

$$\sum_{k=1}^\infty v_{\gamma rk} p_k = \mu_r \ (r = 0, 1, 2 \ldots) \quad (9)$$

Thus, the problem of inverting a characteristic function was formally reduced to a fairly simple mathematical problem - solving a system of linear algebraic equations. However, from a practical point of view, the situation is not so simple. First, the system of equations (9) is infinite-dimensional. For its practical solution, it is necessary to limit its order, that is, to keep in formula (2) a finite number of terms "n". However, even in this case, a situation is quite real when the system of equations (9) turns out to be poorly conditioned, that is, its main determinant turns out to be close to or equal to zero. This situation will occur if the subspace L formed by the vector system $\vec{v}_{\gamma k}$ does not have the property of completeness. The incompleteness of the subspace L means the existence of such vectors of moments $\vec{V}$ in the space R_n, where n is the dimension of the vector of moments along which it is assumed to ensure the coincidence of the initial characteristic function and the approximating distribution (2), which cannot be obtained as a linear combination of vectors of the subspace L, that is, vectors $\vec{v}_{\gamma k}$ $(k = 1, 2 \ldots n)$.

To avoid such a dead end situation, it is necessary to reformulate the problem of constructing an approximating distribution (2) as some kind of approximation problem in the space of moments. To this end, we consider n vectors $\vec{v}_{\gamma k}$ $(k = 1, 2 \ldots n)$ and construct an orthogonal basis consisting of a set of pairwise orthogonal vectors. Here, the orthogonality of vectors is understood in the traditional sense: n-dimensional vectors $\vec{X}$ and $\vec{Y}$ are considered orthogonal if their dot product is zero, that is

$$(\vec{X}, \vec{Y}) = \sum_{k=1}^n X_k Y_k = 0 \quad (10)$$

The orthogonal basis is constructed using the well-known process of orthogonalization in linear algebra and functional analysis.

Let's say

$$\vec{w}_1 = \vec{v}_{\gamma 1} \quad (11)$$

Next, we calculate the coefficient in a linear combination

$$\vec{w}_2 = \mu_{21}\vec{v}_{\gamma 1} + \vec{v}_{\gamma 2} \quad (12)$$

from the condition of orthogonality of vectors $\vec{w}_1$ and $\vec{w}_2$, that is

$$(\vec{w}_2, \vec{w}_1) = \mu_{21}(\vec{v}_{\gamma 1}, \vec{v}_{\gamma 1}) + (\vec{v}_{\gamma 1}, \vec{v}_{\gamma 2}) = 0 \quad (13)$$

Where from

$$\mu_{21} = -\frac{(\vec{v}_{\gamma 1}, \vec{v}_{\gamma 2})}{(\vec{v}_{\gamma 1}, \vec{v}_\gamma)} \quad (14)$$

The further process of orthogonalization of the basis vectors is based on the recurrent formula

$$\vec{w}_{\gamma k} = \sum_{v=1}^k \lambda_{kv}\vec{v}_{\gamma v} \ (k = 1, 2 \ldots), \quad (15)$$

where the coefficients $\mu_{v1}, \mu_{v2} \ldots \mu_{v,v-1}$ are determined from the orthogonality conditions of the vector set of vectors $\vec{w}_{\gamma i}$ $(j = 1, 2 \ldots v - 1)$. We have $(\vec{w}_{\gamma v}, \vec{w}_{\gamma i}) = (\vec{v}_{\gamma v}, \vec{v}_{\gamma i}) + \sum_{j=1}^{v-1} \mu_{vj}(\vec{w}_{\gamma v}, \vec{w}_{\gamma i}) = 0$ (j=1, 2…$v - 1$), from where, taking into account the orthogonality of the vectors $\vec{w}_{\gamma i}$ and $\vec{w}_{\gamma j}$ for all $i, j < v$ and $j \neq i$, we get

$$\mu_{vj} = -\frac{(\vec{v}_{\gamma v}, \vec{w}_{\gamma j})}{(\vec{w}_{\gamma j}, \vec{w}_{\gamma j})} \ (j = 1, 2 \dots v - 1) \tag{16}$$

The described process of orthogonalization of the basis of vectors $\vec{v}_{\gamma k}$ $(k = 1, 2 \dots n)$ allowed us to construct an orthogonal basis $\vec{w}_{\gamma k}$ $(k = 1, 2 \dots n_1)$ isomorphic to the original basis *L*. Generally speaking, the number of vectors of the orthogonal basis may not coincide with the number of vectors in the original basis *L*. In the special case, when the vectors forming the initial basis *L* are linearly independent, the equality $n_1 = n$ holds. Otherwise , the equality $n_1 < n$ is valid , that is , the dimension of the orthogonal basis is less than the number of vectors in the original basis *L*.

The orthogonal basis L_0 has the same remarkable property that an arbitrary vector $\vec{V}_0$ in n-dimensional space R_n can be optimally approximated by the following combination of vectors of the orthogonal basis:

$$\hat{\vec{V}} = \sum_{k=1}^{n_1} p_k \vec{w}_{\lambda k}, \tag{17}$$

where $p_k = \frac{(\vec{V}_0, \vec{w}_{\gamma k})}{(\vec{w}_{\gamma k}, \vec{w}_{\lambda k})}$ $(k = 1, 2 \dots n_1)$.

The approximation $\hat{\vec{V}}$ of the vector $\vec{V}$ found by formula (17) is the best in the sense of minimizing the deviation from the norm of the n-dimensional space R_n. Assuming that procedure (11)-(16) establishes a correspondence between the elements of the initial basis *L* and the elements of the orthogonal basis L_0, we establish an explicit relationship between the vectors of the orthogonal basis L_0 and the initial basis *L*.

3 RESULTS AND DISCUSSIONS

An analysis of the recurrence relations of the orthogonalization process suggests that the desired relationship has the form

$$\vec{w}_{\gamma k} = \sum_{v=1}^{k} \lambda_{kv} \vec{v}_{\gamma v} \ (k = 1, 2 \dots) \tag{18}$$

Let's construct recurrence relations describing the change in weight factors λ_{kv} as the basis expands.

Let the orthogonalization procedure be performed for k vectors included in the basis *L*. Then the expansion of adding a vector to the basis $\vec{v}_{y,k+1}$ means the following transformation of the orthogonal basis: another vector of the form is added to the existing vectors $\vec{w}_{\gamma v}$ $(v = 1, 2 \dots k)$

$$\vec{w}_{\gamma,k+1} = \vec{v}_{\gamma,k+1} + \sum_{j=1}^{k} \mu_{vj} \vec{w}_{\gamma j} \tag{19}$$

After substituting formulas (18) into expression (19), we arrive at the following relation:

$$\vec{w}_{\gamma,k+1} = \vec{v}_{\gamma,k+1} + \sum_{j=1}^{k} \mu_{vj} \sum_{j=1}^{k} \lambda_{j\eta} \vec{v}_{\gamma\eta} \quad or$$

$$\vec{w}_{\gamma,k+1} = \vec{v}_{\gamma,k+1} + \sum_{j=1}^{k} \sum_{\eta=1}^{k} \mu_{kj} \lambda_{j\eta} \vec{v}_{\gamma\eta} \tag{20}$$

After changing the order of summation, we come to the formula

$$\vec{w}_{\gamma,k+1} = \vec{v}_{\gamma,k+1} + \sum_{\eta=1}^{k} \sum_{k=1}^{k} \mu_{kj} \lambda_{j\eta} \vec{v}_{\gamma\eta} \tag{21}$$

Comparing formulas (18) and (21), we arrive at the following recurrence relation:

$$\lambda_{k+1,v} = \begin{cases} \sum_{j=1}^{k} \sum_{\eta=1}^{k} \mu_{vj} \lambda_{j\eta} \ if \ v \le k \\ 1 \ if \ v = k + 1 \end{cases} \tag{22}$$

where the coefficients μ_{vj} $(j = 1, 2 \dots v - 1)$ are calculated using formulas (16). To consistently calculate the coefficients λ_{kv} using formulas (22), their initial values from formula (12) should be taken into account, that is $\lambda_{21} = \mu_{21}, \lambda_{22} = 1$, where μ_{21} it is calculated by the formula (14).

If we take approximation (17) as some best combination of vectors of the orthogonal basis in the space of moments of distributions, then the correspondence of vectors of the orthogonal basis with the initial one established by formula (18) allows $\{\vec{v}_{\gamma k}$ $(k = 1, 2 \dots n\}$ to write the achieved result in terms of the initial basis

$$\hat{\vec{V}}_0 = \sum_{k=1}^{n_1} \rho_k \vec{w}_{\gamma k} = \sum_{k=1}^{n_1} \rho_k \sum_{v=1}^{k} \lambda_{kv} \vec{v}_{\gamma v} \tag{23}$$

Changing the order of summation in the formula, we transform expression (23) to the form

$$\hat{\vec{V}}_0 = \sum_{k=1}^{n_1} \rho_k \vec{w}_{\gamma k} = \sum_{v=1}^{n_1} \sum_{k=v}^{n_1} \rho_k \lambda_{kv} \vec{v}_{\gamma v} \tag{24}$$

In formula (24), we introduce the notation

$$a_v = \sum_{k=v}^{n_1} \rho_k \lambda_{kv} \ (v = 1, 2 \dots n_1) \tag{25}$$

Now formula (24) takes the form

$$\hat{\vec{V}}_0 = \sum_{v=1}^{n_1} a_v \vec{v}_{\gamma v} \tag{26}$$

A comparison of formulas (17) and (26) allows us to conclude that the weighting coefficients p_v $(v = 1, 2 \dots)$ introduced above and the parameters a_v $(v =$

$1, 2 \ldots$) determined by formulas (25) are identical. Thus, the unknown parameters p_v $(v = 1, 2 \ldots n_1)$ in the expression of the approximating function (2) are determined by the formulas

$$p_v = \sum_{k=v}^{n} \rho_k \lambda_{kv} \;\; (v = 1, 2 \ldots n_1) \qquad (27)$$

4 CONCLUSIONS

When solving the problems of risk insurance, the best solution is to use characteristic functions. This is due to their ability to simplify complex operations. First, the summation of independent random variables, which is a cumbersome convolution process when working with distributions, reduces to simple multiplication in terms of characteristic functions. Secondly, taking into account the discount factor necessary to bring future payments to a single value is greatly facilitated by using characteristic functions, unlike traditional distribution functions.

The derived mathematical function is particularly effective in calculating life insurance, where discounting plays an important role. Life insurance gives a person financial security and confidence in the future, so this service will always be in demand and relevant.

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